preprint -

Local ζ-function techniques vs point-splitting procedure: a few rigorous results

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Abstract: Some general properties of local ζ-function procedures to renormalize some quantities in D-dimensional (Euclidean) Quantum Field Theory in curved background are rigorously discussed for positive scalar operators $-\Delta + V(x)$ in general closed D-manifolds, and a few comments are given for nonclosed manifolds too. A general comparison is carried out with respect to the more known point-splitting procedure concerning the effective Lagrangian and the field fluctuations. It is proven that, for $D > 1$, the local ζ-function and point-splitting approaches lead essentially to the same results apart from some differences in the subtraction procedure of the Hadamard divergences. It is found that the ζ function procedure picks out a particular term $w_0(x,y)$ in the Hadamard expansion. The presence of an untrivial kernel of the operator $-\Delta + V(x)$ may produce some differences between the two analyzed approaches. Finally, a formal identity concerning the field fluctuations, used by physicists, is discussed and proven within the local ζ-function approach. This is done also to reply to recent criticism against ζ function techniques.

Introduction.

The ζ function techniques to regularize the determinant of elliptic operators were introduced in Quantum Field Theory by J. S. Dowker and R. Critchley in 1976 [DC76] and S.W. Hawking in 1977 [Ha77]. Since the appearance of these papers, a large use of these techniques has been done by physicists, in particular, to compute one-loop partition functions within semiclassical approaches to the Quantum Gravity and also concerning other related areas [BCVZ96, EORBZ94, El95, Ca90]. After the fundamental works cited above, many efforts have been spent in studying the black hole entropy and related problems by these approaches (for recent results see [CKVZ, MI97, IM96]).
The *local* ζ-function approach differs from integrated ζ function approaches because the former defines quantities which may depend on the point on the manifold and thus can be compared

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with analogue regularized and renormalized quantities produced from different more usual local approaches as the point-splitting one [BD82, Fu91]. A first step toward these approaches was given by R. M. Wald [Wa79] who considered explicitly a local ζ function to regularize and renormalize the local effective Lagrangian of a field operator $-\Delta + m^2$. Similar results, actually in a very formal fashion, paying attention to the physical meaning rather than the mathematical rigour, have been obtained successively for operators $-\Delta + m^2 + \xi R$ (also considering higher spin) employing the so-called "Schwinger-DeWitt expansion". Within Wald's paper, is conjectured that a method based on a local ζ function should exist also for the stress tensor and the results obtained from this method should agree with the results obtained via point-splitting approaches. A direct rigorous prove of this agreement is contained in the same paper concerning the effective Lagrangian in four dimension and for a motion operator "Laplacian plus squared mass". Formal proofs concerning the effective Lagrangian in more general cases can be found in [BD82]. Recently, the local approaches have proven to give untrivial results if compared with global approaches based on heat kernel procedures [ZCV96, IM96, MI97] whenever the manifold contains singularities physically relevant. Just as conjectured by Wald, methods based on local ζ function have been found out which are able to regularize and renormalize the one-loop averaged squared field fluctuations [IM98] and the one-loop stress tensor [Mo97a] directly (anyhow, a first remarkable formal attempt still appeared in [Ha77]). In all examined cases, agreement with the point-splitting technique has arisen as well as, sometimes, differences with integrated heat kernel approaches.

This paper is devoted to study the relation between *local* ζ function approaches and point-splitting techniques in deep and within a rigorous mathematical framework. For this task, in the first section we shall review the basic concepts of the heat kernel theory for an operator "Laplacian plus potential" in a closed D dimensional manifold and review some issues related to the Euclidean path integral. In the second part we shall consider the relationship between heat-kernel, *local* ζ function and some physical quantities related to the Euclidean functional integral in compact curved manifolds. We shall study also the relation between the point-splitting procedure and local ζ-function approach concerning the effective Lagrangian and the field fluctuations. In particular, we shall prove that the "Green function" generated by the local ζ function (defined also when the operator A^{-1} does not exist) has the Hadamard short-distance behaviour for any dimension $D > 1$. We shall prove that the point-splitting approach gives the same results of the local ζ function technique, apart from a different freedom/ambiguity in choosing a particular term in the Hadamard expansion of the Green function of the operator A. We shall give also some comments either for the case of the presence of boundary or a noncompact manifold. In the end, we shall prove that some identities supposed true by physicists, concerning the field fluctuations and two-point functions, can be regularized and rigorously proven within the local ζ function approach. (We shall find also a simple application of Wodzicki's residue and Connes' formula.) This will be done also to reply to a recent criticism against the ζ function techniques [Ev98] where it is erroneously argued that similar formal properties do not work within the ζ function approach.

1 Preliminaries.

Within this section, we summarize some elementary concepts related to the heat-kernel necessary to develop the theory of the *local* ζ function for differential operators in QFT in curved background. For sake of brevity, known theorems or theorems trivially generalizable from known result, will be given without explicit proof. References and several comments concerning these theorems will be anyhow supplied. We shall consider almost only compact manifolds. General references including topics on ζ function techniques are [Ch84] and [Sh87, Da89] which use "geometrical" and more "analytic" approaches respectively. A very general treatise concerning also pseudodifferential operators is [Gi84].

1.1 General hypotheses.

Throughout this paper, $\mathcal{M}$ is a Hausdorff, connected, oriented, C^∞ Riemannian D-dimensional manifold. The metric is indicated with g_{ab} in local coordinates. We suppose also that $\mathcal{M}$ is compact without boundary (namely is "closed"). We shall consider real elliptic differential operators in the Schrödinger form "Laplace-Beltrami operator plus potential"

$$A' = -\Delta + V \;\; : \;\; C^\infty(M) \to L^2(\mathcal{M}, d\mu_g) \tag{1}$$

where, locally, $\Delta = \nabla_a \nabla^a$, and ∇ means the covariant derivative associated to the metric connection, $d\mu_g$ is the Borel measure induced by the metric, and V is a *real* function belonging to $C^\infty(\mathcal{M})$. We assume also that A' is bounded below by some $C \geq 0$.
All the requirements above on both $\mathcal{M}$ and A' are the **general hypotheses** which we shall refer to throughout this paper.

A countable base of the topology is required in order to endow the manifold with a partition of the unity and make $L^2(\mathcal{M}, d\mu_g)$ separable. This allows the use of the integral representation Hilbert-Schmidt operators. In our case, the requested topological property follows from the compactness and the Hausdorff property.

Concerning the operators A' defined above, we notice that they are symmetric on $C^\infty(\mathcal{M})$ and admit self-adjoint extensions since they commute with the anti-unitary complex-conjugation operator in $L^2(\mathcal{M}, d\mu_g)$ [RS]. In particular, one may consider the so-called Friedrichs self-adjoint extension A of A' [RS] which, as is well known, is bounded from below by the same bound of A'. A sufficient conditions which assure $A' \geq 0$, (see Theorem 4.2.1 in [Da89]) is the existence of a strictly positive $C^2(\mathcal{M})$ function ϕ such that, everywhere,

$$\phi(x)V(x) - \nabla_a \nabla^a \phi(x) \geq 0 \tag{2}$$

In this case A' is bounded below by $C = \inf_{x \in \mathcal{M}} \{V(x) - \nabla_a \nabla^a \phi(x)/\phi(x)\} \geq 0$. Notice that this condition may hold also for V nonpositive and the simplest sufficient condition, $V(x) \geq 0$ everywhere, is a trivial subcase, as well.

Concerning **Theorem 1.3** and successive ones, we shall suppose also that the *injectivity radius* r of $\mathcal{M}$ is strictly positive. The *injectivity radius* is defined as $r = \inf_{p \in \mathcal{M}} d(p, C_m(p))$ where the *cut locus* $C_m(p)$ is the set of the union of the *cut point* of p along all of geodesics

that start from p, d is the geodesical distance [Ch84, dC92]. The function $(x,y) \mapsto d^2(x,y)$ is everywhere continuous in $\mathcal{M} \times \mathcal{M}$ and, furthermore, the relevant fact for the heat kernel expansion theory is that, for $r > 0$, whenever $d(p,q) < r$, there is a local chart corresponding to a normal coordinate system (the *exponential map*) centered in p (resp. q) which contains also the point q (resp. p). Within this neighborhood, the function $x \mapsto d^2(p,x)$ (resp. $x \mapsto d^2(x,q)$) is also C^∞. Moreover, by this result and employing Sobolev's Lemma [Ru97], one finds that the function $(x,y) \mapsto d^2(x,y)$ belongs to $C^\infty(\{(x,y) \in \mathcal{M} \times \mathcal{M} \mid d(x,y) < r\})$.

Sufficient conditions for having $r > 0$ are found in Chapter 13 of [dC92]. In particular, a strictly positive upper bound K for the sectional curvature of a compact manifold is sufficient to have $r > 0$ since $r \geq \pi/\sqrt{K}$. Notice also that, for instance, a Riemannian manifold symmetric under a Lie group of isometries involves $r > 0$ trivially.

As a general final remarks, we specify that, throughout this paper, "holomorphic" and "analytic" are synonyms and natural units $c = \hbar = 1$ are employed.

1.2 The physical background.

All quantities related to A' we shall consider, for $D = 4$, appear in (Euclidean) QFT in curved background and concern the theory of quasifree scalar fields. In several concrete cases of QFT, the form of $V(x)$ is $m^2 + \xi R(x) + V'(x)$ where m is the mass of the considered field, R is the scalar curvature of the manifold, ξ is a real parameter and V' another smooth function not dependent on g_{ab}. All the physical quantities we shall consider are formally obtained from the Euclidean functional integral

$$Z[A'] := \int \mathcal{D}\phi \, e^{-\frac{1}{2}\int_{\mathcal{M}} \phi A' \phi \, d\mu_g} \, . \tag{3}$$

The integral above can be considered as a partition function of a field in a particular quantum state corresponding to a canonical ensemble. Often, the limit case of vanishing temperature is also considered and in that case the manifold cannot be compact. The direct physical interpretation as a partition function should work provided the manifold has a Lorentzian section obtained by analytically continuing some global temporal coordinate $x^0 = \tau$ into imaginary values $\tau \to it$ and considering (assuming that they exist) the induced continuations of the metric and relevant quantities. It is required also that ∂_τ is a global Killing field of the Riemannian manifold generated by an isometry group S_1, which can be continued into a (generally local) time-like Killing field ∂_t in the Lorentzian section (see [Ha77] and [Wa79]). Then one assumes that $k_B\beta$ is the inverse of the temperature of the canonical ensemble quantum state, β being the period of the coordinate τ. Similar interpretations hold for the (analytic continuations of) quantities we shall introduce shortly. The (thermal) quantum state which all the theory is referred to is determined by the Feynman propagator obtained by analytical continuation of the Green function of the operator A'. For this reason the analysis of the uniqueness of the Green functions of the operator A' is important. A general discussion on these topics, also concerning grand canonical ensemble states can be found in [Ha77].

Physicists, rather than trying to interpret the integral in (3) as a Wiener measure, generalize the trivial finite-dimensional Gaussian integral and re-write the definition of $Z[A']$ as [Ha77]

$$Z[A'] := \left[\det\left(\frac{A'}{\mu^2}\right)\right]^{-1/2} \tag{4}$$

provided a useful definition of the determinant of the operator A' is given. The mass scale μ present in the determinant is necessary for dimensional reasons [Ha77]. Such a scale introduces an ambiguity which remains in the finite renormalization parts of the renormalized quantities and, dealing with the renormalization of the stress tensor within the semiclassical approach to the quantum gravity, it determines the presence of quadratic-curvature terms in effective Einstein's equations [Mo97a]. Similar results are discussed in [Wa94, BD82, Fu91] employing other renormalization procedures (point-splitting).

The theory which we shall summarize in the following has been essentially developed to give a useful interpretation of $\det A'$, anyhow it has been successively developed to study several different, formally quadratic in the field, quantities related to the functional determinant above. Some of these are[2], where the various symbols "=" have to be opportunely interpreted, the *effective action*

$$S[A'] = -\ln Z[A'] = \int_{\mathcal{M}} \mathcal{L}(x|A') d\mu_g(x) \;, \tag{5}$$

where the integrand is the *effective Lagrangian*; the *field fluctuations*

$$\langle \phi^2(x|A') \rangle = Z[A']^{-1} \int \mathcal{D}\phi \, e^{-\frac{1}{2}\int_{\mathcal{M}} \phi A' \phi \, d\mu_g} \; \phi(x)\phi(x) = \frac{\delta}{\delta J(x)}|_{J\equiv 0} S[A' + 2J] \;, \tag{6}$$

and the *one-loop averaged stress tensor*

$$\langle T_{ab}(x|A') \rangle = Z[A']^{-1} \int \mathcal{D}\phi \, e^{-\frac{1}{2}\int_{\mathcal{M}} \phi A' \phi \, d\mu_g} \; T_{ab}(x) = \frac{2}{\sqrt{g}} \frac{\delta}{\delta g_{ab}(x)} S[A'] \;, \tag{7}$$

where $T_{ab}(x)$ is the classical stress-tensor (in a paper in preparation we analyze the stress tensor renormalization.) Recently, some other nonquadratic quantities have been considered in the heat-kernel or ζ-function approaches [PH96].

All quantities in left hand sides of (5), (6), (7) and the corresponding ones in the Lorentzian section are affected by divergences whenever one tries to compute them by trivial procedures [Wa79, BD82, Fu91]. For instance, interpreting the functional integral of $\phi(x)\phi(y)$ in (6) as the Green function of A' (the analytic continuation of the Feynman propagator), $G(x,y)$

$$\langle \phi(x)\phi(y) \rangle = Z[A']^{-1} \int \mathcal{D}\phi \, e^{-\frac{1}{2}\int_{\mathcal{M}} \phi A' \phi \, d\mu_g} \; \phi(x)\phi(y) = G(x,y) \;, \tag{8}$$

the limit $y \to x$, necessary to get $\langle \phi^2(x) \rangle$, diverges as is well known. One is therefore forced to remove *by hand* these divergences, this nothing but the main idea of the *point-splitting procedure*. It is worth remarking that the *definitions* given in terms of ζ function and heat kernel

[2] In defining the effective action and so on, we are employing the opposite sign conventions with respect to [Mo97a], our conventions are the same used in [Wa79].

[Ha77, Wa79, BD82, Mo97a, IM98] of the formal quantity in left hand sides of (5), (6), (7) contain an implicit *infinite renormalization* procedure in the sense that these are finally free from divergences.

1.3 Heat kernel.

The key to proceed with the ζ-function theory in order to provide a useful definition of determinant of the operator (4) A' is based upon the following remark. In the case A is a $n \times n$ Hermitian matrix with eigenvalues $\lambda_1, \lambda_2, \dots \lambda_n$, then (the prime indicates the s-derivative)

$$\det A = \prod_{j=1}^{n} \lambda_j = e^{-\zeta'(0|A)} \tag{9}$$

where, we have defined the ζ-function of A as

$$\zeta(s|A) = \sum_{j=1}^{n} \lambda_j^{-s} \,. \tag{10}$$

The proof of (9) is direct. Therefore, the idea is to generalize (9) to *operators* changing the sum into a series (the spectrum of A' is discrete as we shall see). Unfortunately this series diverges at $s = 0$ as we shall see shortly. Anyhow, it is possible to *continue analytically* $\zeta(s|A')$ in a neighborhood of $s = 0$ and *define* the determinant of A' in terms of the continued function $\zeta(s|A')$. This generalization requires certain well-known mathematical tools and untrivial results we go to summarize.

First of all, let us give the definition of *heat Kernel* for operators A' defined above [Ch84, Da89, Gi84, Wa79, Ha77, Fu91]. Three relevant theorems follows the definition.

Definition 1.1. *Within our general hypotheses on $\mathcal{M}$ and A', let us consider, if it exists, a class of operators*

$$(K_t\psi)(x) := \int_{\mathcal{M}} K(t,x,y|A')\psi(y)\, d\mu_g(y) \quad t \in (0,+\infty) \tag{11}$$

where the integral kernel is required to be $C^0((0,+\infty)\times\mathcal{M}\times\mathcal{M}))$, $C^1((0,+\infty))$ in the variable t, $C^2(\mathcal{M})$ in the variable x and satisfy the "heat equation" with an initial value condition:

$$\left[\frac{d}{dt} + A'_x\right] K(t,x,y|A) = 0 \tag{12}$$

and

$$K(t,x,y|A') \to \delta(x,y) \;\; \text{as } t \to 0^+ \,, \tag{13}$$

the limit is understood in a "distributional" sense, i.e., once the kernel is y-integrated on a test function $\psi = \psi(y)$ belonging to $C^0(\mathcal{M})$ and (13) means

$$\lim_{t\to 0^+} \int_{\mathcal{M}} K(t,x,y|A')\psi(y)\, d\mu_g(y) = \psi(x) \quad \text{for each } x \in \mathcal{M} \tag{14}$$

The kernel $K(t,x,y|A')$, if exists, is called **heat kernel** *of A'.*

Theorem 1.1. *In our general hypotheses on $\mathcal{M}$ and A'*

(a) *the set of the operators K_t above defined exists is unique and consists of self-adjoint, bounded, compact, Hilbert-Schmidt, trace-class operators represented by a $C^\infty((0,+\infty)\times\mathcal{M}\times\mathcal{M})$ integral kernel. This is also real, symmetric in x and y and* positive *provided either V is positive or* (2) *is satisfied for some ϕ and* strictly positive *in the case $V \equiv m^2$(constant) ≥ 0.*

(b) *Moreover*

$$K(t,x,y|A') = \sum_{j=0}^{\infty} e^{-t\lambda_j}\phi_j(x)\phi_j^*(y) \tag{15}$$

where the series converges on $[\alpha+\infty)\times\mathcal{M}\times\mathcal{M}$ absolutely in uniform sense (i.e. the series of the absolute values converges uniformly) for any fixed $\alpha \in (0,+\infty)$,

$$0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \; ... \to +\infty \tag{16}$$

are eigenvalues of A' and $\phi_j \in C^\infty(\mathcal{M})$ are the corresponding normalized eigenvectors, the dimension of each eigenspace being finite.

(c) *The class $\{\phi_j | j = 0,1,2,...\}$ defines also a Hilbertian base of $L^2(\mathcal{M}, d\mu_g)$ and*

$$\int_{\mathcal{M}} d\mu_g(x)\, K(t,x,x|A) = \sum_{j=0}^{+\infty} e^{-\lambda_j t} = TrK_t\,. \tag{17}$$

All these results are straightforward generalizations of theorems contained in Section 1 of Chapter VI of [Ch84], the convergence properties follow from Mercer's theorem [RN80]; see also [Da89] concerning the issue of the positivity of the heat kernel and Sections 3 and 4 of Chapter VI of [Ch84] concerning the existence of the heat kernel under the further hypotheses $r > 0$[3]. The existence of the heat kernel can be proven without this hypothesis by studying the integral kernel of the exponential of the Friedrichs self-adjoint extension of A' as done in Chapter 5 of [Da89]. By Nelson's theorem [RS], using the class of all linear combinations of vectors ϕ_j as a dense set of analytic vectors, one proves that A' is essentially self-adjoint in $C^\infty(\mathcal{M})$; this leads to

Theorem 1.2. *In our general hypotheses on M and A' defined on the domain $C^\infty(\mathcal{M})$,*

(a) *there is only one self-adjoint extension of A', namely, its closure $\bar{A}'$, which also coincides with the Friedrichs self-adjoint extension of A', A;*

(b) *this extension is bounded below by the same bound of A' and*

$$\sigma(A) = \sigma_{\text{p.p.}}(A) = \sigma_{\text{disc.}}(A) = \{\lambda_n | n = 0,1,2...\}\;;$$

[3]The reader has to handle with great care the content of [Ch84] since, unfortunately, some missprints appear in several statements. For instance, (45) in Section 4 Chapter VI is incorrect due to the presence of the operator L_x, this is the reason for the introduction of the parameter η in our **Theorem 1.3**. Moreover, **Lemma 2** in [Ch84] requires $F \in C^1$ rather than $F \in C^0$ as erroneously written there.

(c) *in the usual spectral-theory sense, for* $t \in (0, +\infty)$

$$K_t = e^{-tA} \tag{18}$$

and $\{K_t | t \in (0, +\infty)\}$ *is a strongly continuous one-parameter semigroup of bounded operators. In particular* $K_t \to I$ *as* $t \to 0^+$ *in the strong operator topology and* $K_t \to P_0$ *as* $t \to +\infty$ *(*P_0 *being the projector onto* $Ker\ A$*) in the strong operator topology. The limit above holds also in the sense of the uniform punctual convergence whenever* K_t *acts on* $\psi \in C^0(\mathcal{M})$.

From now on, since A' determines A uniquely and $A' = A|_{C^\infty(\mathcal{M})}$, we shall omit the prime on A' almost everywhere.

Generalizing the content of Section 3 and 4 of Chapter VI of [Ch84] [4] one also gets a well-known "asymptotic expansion" of the heat kernel for $t \to 0^+$ [Ch84, Fu91, Ca90].

Theorem 1.3. *In our general hypotheses on* $\mathcal{M}$ *and* A'*, supposing also that* $r > 0$*,*

(a) *for any fixed integer* $N > D/2 + 2$ *and any fixed real* $\eta \in (0, 1)$ *it holds*

$$K(t, x, y|A) = \frac{e^{-\sigma(x,y)/2t}}{(4\pi t)^{D/2}} \chi(\sigma(x, y)) \sum_{j=0}^{N} a_j(x, y|A) t^j + \frac{e^{-\eta\sigma(x,y)/2t}}{(4\pi t)^{D/2}} t^N O_\eta(t; x, y) \,, \tag{19}$$

where
(1) $2\sigma(x, y) = d^2(x, y)$, $\chi = \chi(u)$ *is a non-negative function in* $C^\infty([0, +\infty))$ *which takes the constant value* 1 *for* $|u| < r^2/16$ *and vanishes for* $|u| \geq r^2/4$.
(2) O_η *is a function in* $C^0([0, +\infty) \times \mathcal{M} \times \mathcal{M})$ *at least, and it is such that the function*

$$(t, x, y) \mapsto \frac{e^{-\eta\sigma(x,y)/2t}}{(4\pi t)^{D/2}} O_\eta(t, x, y) \tag{20}$$

belongs to $C^\infty((0, +\infty) \times \mathcal{M} \times \mathcal{M})$*. Moreover, for any positive constant* U_η *and for* $0 \leq t < U_\eta$ $|O_\eta(t, x, y)| < B_\eta t$ *holds true for a corresponding positive constant* B_η*, not depending on* x *and* y *in* $\mathcal{M}$.
(3) *The coefficients* $a_j(x, y|A)$ *are defined when* x *and* y *belong to* $\{(x, y) \in \mathcal{M} \times \mathcal{M} \,|\, d(x, y) < r\}$ *and are* C^∞ *therein, in particular* $\chi(\sigma) a_j \in C^\infty(\mathcal{M} \times \mathcal{M})$ $(j = 0, 1, 2...)$.

(b) *The* $C^\infty((0, +\infty) \times \mathcal{M} \times \mathcal{M})$ *functions called* parametrices

$$F_N(t, x, y) = \frac{e^{-\sigma(x,y)/2t}}{(4\pi t)^{D/2}} \chi(\sigma(x, y)) \sum_{j=0}^{N} a_j(x, y|A) t^j \quad N = 0, 1, 2, \ ... \tag{21}$$

for each $N = 0, 1, 2, ...$ *fixed, satisfy, working as integral kernel on functions in* $C^0(\mathcal{M})$*,*

$$F_N(t, x, y) \to \delta(x, y) \text{ as } t \to 0^+ \,. \tag{22}$$

Notice that, the values $r^2/4$ and $r^2/16$ in the definition of χ may be changed, their task is just to make everywhere C^∞ the right hand side of (19) also when x is too far from y. Concerning

[4] See the previous footnote.

the precise form of the coefficients a_j, we have that all $a_j(x,y|A)$ can be obtained [Ca90] by canceling both χ and O_η out and *formally* substituting the expansion (19) with $N = +\infty$ (this limit usually does *not* exist) into the heat kernel equation (12) and requiring that the coefficients of each t^j vanish separately. This produces the set of recurrent differential equations in each normal convex neighborhood $\mathcal{N}_y$ centered in y and referred to spherical coordinates $(x^a)_{a=1,\ldots,D} = (\rho,\Omega) \equiv x$ (ρ is the geodesical distance of x from y) [Ch84, Ca90]

$$\rho\partial_\rho\left(a_0(x,y|A)\Delta^{-1/2}_{VVM}(x,y)\right) = 0\,, \tag{23}$$

$$\begin{aligned} -\Delta^{-1/2}_{VVM}(x,y)\, A'_x a_j(x,y|A) &= \rho\partial_\rho\left(a_{j+1}(x,y|A)\,\Delta^{-1/2}_{VVM}(x,y)\right) \\ &+ (j+1)\,a_{j+1}(x,y|A)\,\Delta^{-1/2}_{VVM}(x,y) \quad (j>0)\,, \end{aligned} \tag{24}$$

$\Delta_{VVM}(x,y)$ is defined below. These equations determine uniquely the coefficients $a_j(x,y|A)$ once one fixes $a_0(x,y|A)$ and requires that $a_j(x,y|A)$ $(j = 0,1,2\ldots)$ is bounded as $x \to y$.
To assure the validity of (22), it is sufficient to requires that $a_0(x,y|A)$ which satisfies (23) is smooth and $a_0(x,y|A) \to 1$ as $x \to y$ not depending on Ω. Then (23) and $\Delta_{VVM}(y,y) = 1$ imply

$$a_0(x,y|A) = \Delta^{1/2}_{VVM}(x,y) \tag{25}$$

Δ_{VVM} is the bi-scalar called Van Vleck-Morette determinant. It defines the Riemannian measure in normal Riemannian coordinates $x^a \equiv x$ in $\mathcal{N}_y$ where $\Delta_{VVM}(x,y)^{-1} = \sqrt{g(x^a)}$ (and thus $\Delta_{VVM}(y,y) = 1$) [Ca90]. $\Delta_{VVM}(x,y) = -[g(x)g(y)]^{-1/2}\det\{\partial^2\sigma(x,y)/\partial x^a\partial y^b\}$ holds in general coordinates. The mass dimensions of the coefficients $a_j(x,y|A)$ and the other relevant quantities are $[t] = M^{-2}$, $[a_j(x,y|A)] = M^{2j}$, $[A] = M^2$ and $[K(t,x,y|A)] = M^D$.

The expansion we have presented here (see also [Fu91, Ca90, BCVZ96]) is a bit different from the more usual Schwinger-De Witt one [BD82, Wa79], where a further overall exponential $\exp -tm^2$ appears in the right hand side of the expansion of $K(t,x,y|A)$, m being the mass of the field. The Schwinger-De Witt coefficients are related to those in (19) by trivial relations [Ca90]. The explicit expression of some of the coefficients $a_j(x,y|A)$ (also for the Schwinger-DeWitt expansion and Lorentzian metrics) for $x = y$ and $x \neq y$ can be found in [BCVZ96, KK98] and [BD82] respectively. Apart from terms depending on the particular form of V, they are polynomials in the curvature tensors of the manifold.

Theorem 1.4. *In our general hypotheses on $\mathcal{M}$ and A' and $r > 0$ we have Weyl's formula*

$$\lim_{j\to+\infty}\frac{\lambda_j^{D/2}}{j} = \frac{(2\pi)^D}{\omega_D V(\mathcal{M})}\,, \tag{26}$$

ω_D is the volume of the unit disk in $I\!R^D$, $\omega_D = \pi^{D/2}/\Gamma(1+\ D/2)$.

2 Local ζ-function techniques and point-splitting procedure.

In this section we develop the theory of the *local* ζ function and then, coming back to physics, we shall consider how this theory is employed. In particular, proving some rigorous theorem concerning the local ζ function approach to define and regularize the physical quantities $\det A$ (namely $S[A]$), $\mathcal{L}(x|A)$, $\langle\phi^2(x|A)\rangle$ given in **1.2** and their relations with the corresponding point-splitting procedures. The case of $\langle T_{ab}(x|A)\rangle$ will be treated in a paper in preparation.

References concerning the physical applications are respectively [Ha77, Wa79, BD82] concerning the effective Lagrangian (and effective action), [MI97] concerning the field fluctuations and [Mo97a] concerning the averaged one-loop stress tensor. Further references on the heat-kernel and ζ-function techniques in symmetric manifolds are [Ca90, EORBZ94].

2.1 The local ζ function.

Definition 2.1 *Within our initial hypotheses on $\mathcal{M}$ and A', and $r > 0$, the* **local "off-diagonal" ζ function** *of A is the function defined for Re $s > D/2$, $x, y \in \mathcal{M}$ and $\mu^2 > 0$*

$$\zeta(s, x, y|A/\mu^2) := \frac{1}{\Gamma(s)} \int_0^{+\infty} d(\mu^2 t)(\mu^2 t)^{s-1} \left[K(t, x, y|A) - P_0(x, y|A)\right] . \tag{27}$$

The mass-square parameter μ^2 (almost always omitted at this step) is actually necessary from dimensional considerations and it is not fixed from the theory as remarked above. $P_0(x, y|A)$ is the integral kernel of the projector onto the kernel of A. The given definition is well-posed (as proven within the proof of **Theorem 2.2** below) since the integral above converges absolutely for $Re\ s > D/2$ essentially because of the exponential decay of $K - P_0$ at large t and the expansion (19) as $t \to 0^+$ which fixes the bound $Re\ s > D/2$.

All that follows (essentially based on theorems by Minakshisundaram and Pleijel [Ch84]) is a direct consequence of the heat kernel expansion (19), (15), Weyl's asymptotic formula (26) (which trivially implies that the series of λ^{-s} converges for $Re\ s > D/2$ and diverges for $Re\ s < D/2$)) and the well-known identity for $a > 0$, $s \in \mathbb{C}$, $Re\ s > 0$

$$a^{-s}\Gamma(s) = \int_0^{+\infty} dt\, t^{s-1}\, e^{-ta} . \tag{28}$$

The properties of uniform and absolute convergence are, once again, consequences of Mercer's theorem [RN80]. In particular, the following theorem can be proven by generalizing the the content of Remark 2 in Chapter VI, Section 4 of [Ch84]. Anyhow, the first and the last statement will be proven within the proof of **Theorem 2.2** below.

Theorem 2.1. *In our general hypotheses on $\mathcal{M}$ and A' and $r > 0$, for $\mu^2 > 0$ and Re $s > D/2$,*

(a) *the integral in* (27) *converges absolutely;*

(b) *for s fixed in the region given above, the function of x and y, $\zeta(s, x, y|A/\mu^2)$ is an integral kernel of the bounded trace-class operator $(A/\mu^2)^{-s}$ defined by spectral theory in the usual way, through a projector valued measure (dropping the spectral-measure part on the kernel of A whenever it exists);*

(c) *for $Re\ s > D/2$, the prime meaning that any possible vanishing eigenvalues and corresponding eigenvectors are omitted from the sum,*

$$\zeta(s, x, y|A/\mu^2) = \sum_{j\in I\!N}{}' \left(\frac{\lambda_j}{\mu^2}\right)^{-s} \phi_j(x)\phi_j^*(y) \; , \tag{29}$$

where the convergence is absolute in uniform sense in $\{s \in \mathbb{C} \mid \beta \geq Re\ s \geq \alpha\} \times \mathcal{M} \times \mathcal{M}$ for any couple $\alpha, \beta \in I\!R$ with $\beta > \alpha > D/2$, and thus $\zeta(s, x, y|A/\mu^2)$ defines a s-analytic function in $C^0(\{s \in \mathbb{C} \mid Re\ s > D/2\} \times \mathcal{M} \times \mathcal{M})$.

We remark that $\zeta(s, x, y|A/\mu^2)$, for $Re\ s > D/2$, could be defined by (29) with no reference to the heat kernel, obtaining the same results.

Definition 2.2. *Within our general hypotheses on $\mathcal{M}$ and A', and $r > 0$, the* **local ζ function** *of the operator A is the function of $x \in \mathcal{M}$ and $s \in \mathbb{C}$ with $Re\ s > D/2$, $\mu^2 > 0$*

$$\zeta(s, x|A/\mu^2) := \zeta(s, x, x|A/\mu^2) \; . \tag{30}$$

Similarly, the **"integrated" ζ function** *of A, $\zeta(s|A/\mu^2)$ is defined by x integrating the local one for $Re\ s > D/2$*

$$\zeta(s|A/\mu^2) := \int_{\mathcal{M}} d\mu_g(x)\zeta(s, x|A/\mu^2) \tag{31}$$

Notice that we have from (29), because of the uniform convergence, for $Re\ s > D/2$,

$$\zeta(s|A/\mu^2) = \sum_{j\in I\!N}{}' \left(\frac{\lambda_j}{\mu^2}\right)^{-s} = \mathrm{Tr}\left[\left(\frac{A}{\mu^2}\right)^{-s}\right] \tag{32}$$

The operator A^{-s} is defined via spectral theory omitting the the spectral-measure part corresponding to the kernel of A whenever the kernel is not trivial. As in the case of the local ζ function, this series (which *diverges* for $Re\ s < D/2$) converges absolutely in uniform sense for $s \in \{z \in \mathbb{C} \mid \beta \geq Re\ z \geq \alpha\}$, for $\beta > \alpha > D/2$.

We are now able to state and prove the most important theorem on the local ζ function.

Theorem 2.2. *Let us suppose $\mathcal{M}$ and A' satisfy our general hypotheses and also $r > 0$ and $\mu^2 > 0$.*

(a) *Whenever $x \neq y$ are fixed in $\mathcal{M}$,*

(1) *$\zeta(s, x, y|A/\mu^2)$ can be analytically continued in the whole s-complex plane defining an everywhere holomorphic function of s which still satisfies* (27) *for $Re\ s > 0$; moreover, this holds everywhere in $s \in \mathbb{C}$ provided $P_0 \equiv 0$.*

(2) *The function obtained by varying s, x, y belong at to $C^0(\mathbb{C} \times ((\mathcal{M} \times \mathcal{M}) - \mathcal{D}))$ together with all of its s derivative, where $\mathcal{D} := \{(x, y) \in \mathcal{M} \times \mathcal{M} \mid x = y\}$.*

(b) *Whenever $x = y$ are fixed in $\mathcal{M}$,*

(1) *$\zeta(s, x, x|A/\mu^2)$ can be analytically continued in the variable s into a meromorphic function with possible poles, which are simple poles, situated in the points*

$$\begin{aligned} s_j &= D/2 - j, \quad j = 0, 1, 2, \dots && \text{if } D \text{ is odd, or} \\ s_j &= D/2 - j, \quad j = 0, 1, 2, \dots D/2 - 1 && \text{if } D \text{ is even} \end{aligned}$$

and residues

$$Res(s_j) = \frac{\mu^{D-2j} a_j(x, x|A)}{(4\pi)^{D/2}\Gamma(D/2 - j)} \,. \tag{33}$$

(2) *Varying s and x, one gets a function which belongs to $C^0((\mathbb{C} - \mathcal{P}) \times \mathcal{M})$ together with all of its s derivatives, where $\mathcal{P}$ is the set of the actual poles (each for some x) among the points listed above.*

(c) *For x, y fixed in $\mathcal{M}$, the s-continued function $\zeta(s, x, y|A/\mu^2)$ is analytic in a neighborhood of $s = 0$ and*

$$\zeta(0, x, y|A/\mu^2) + P_0(x, y|A) = \frac{a_{D/2}(x, x|A)}{(4\pi)^{D/2}} \, \delta_{x,y} \, \delta_D \tag{34}$$

where $\delta_{x,y} = 1$ if $x = y$ and $\delta_{x,y} = 0$ otherwise, $\delta_D = 1$ if D is even and $\delta_D = 0$ if D is odd. For $x \neq y$ the zero at $s = 0$ of right hand side of (34) *is of order ≥ 1.*

(d) *The analytic continuation of the integrated ζ function coincides with the integral of the analytic continuation of the local (*on-diagonal*) ζ function and has the same meromorphic structure of the continued function $\zeta(s, x|A/\mu^2)$ with possible poles on the same points and residues given by the integrals of the residues* (33).

Proof. See **Appendix** □.

Comments.

(1) It is worth stressing that, whenever A has no vanishing eigenvalue (so that $P_0 \equiv 0$) and $x \neq y$ or, equivalently, whenever $Re\ s > 0$ and $x \neq y$, the relation (27) is maintained also when the left hand side is understood in the sense of the analytic continuation and the right hand side is computed as a proper integral. This property will be very useful studying the Green function of A.

(2) The simple poles of the local or integrated ζ function are related to the heat kernel coefficients in a direct way as follows from (33). It is very important to stress that there is no guarantee that, actually, poles appear in the points indicated above because the corresponding coefficients $a_j(x, x|A)$ (or the integrated ones) may vanish. Anyhow, *if* poles appear, they must belong to the sets listed above.

(3) As a final comment we remark that (34) proves that the continued function $\zeta(s, x, y|A/\mu^2)$

is not continuous on the diagonal $x = y$, at least for $s = 0$. So the s-continuation procedure and the limit as $x \to y$ *generally do not commute.*

Remark. From now on, barring different specification, the symbols of the various ζ functions as $\zeta(s, x, y|A/\mu^2)$ indicate the meromorphic functions continued from the initial domain of definition $Re\ s > D/2$ as far as possible in the complex s plane.

We are now able to define in a mathematical precise meaning within the framework of the ζ function theory the determinant of A necessary in (4).

Definition 2.3. *Within our general hypotheses on $\mathcal{M}$ and A', for $r > 0$ and $\mu^2 > 0$, the* **determinant** *of the operator A/μ^2 is defined as*

$$\det\left(\frac{A}{\mu^2}\right) := e^{-\frac{d}{ds}|_{s=0}\zeta(s|A/\mu^2)} . \tag{35}$$

2.2 A few comments in more general cases.

What is it maintained of these results once one drops the hypotheses of a compact manifold and/or absence of boundaries? More general results of the heat kernel theory for the pure Laplacian, with trivial extensions to the case $A = -\nabla_a\nabla^a + m^2$ can be found in the literature (see [Wa79, Ch84, Da89]). A general discussion on the heat kernel, considering also vectorial and tensorial fields and more general connections than the metrical one, can be found in [Fu91].

In general, the lack of the hypothesis of a compact manifolds produces the failure of expansions as those in (15) because the spectrum of A, the Friedrichs self-adjoint extension of A' (which has to be defined on $C_0^\infty(\mathcal{M})$ and still results to be essentially self-adjoint [Da89]) becomes continuous in general. One sees that, in particular cases, it is possible to replace the sum in (15) with integrals dealing with opportune spectral measures

$$K(t, x, y|A) = \int_{\sigma(A)} d\mu_A(\lambda) \sum_j e^{-\lambda t}\phi_{j\lambda}(x)\phi^*_{j\lambda}(y) , \tag{36}$$

the function $\phi_{j\lambda}$ being eigenfunctions (in some weak sense) of A with eigenvalue λ.
We expect that this is a general result. This can be done also for the local ζ function, which can be still defined by (27) provided the corresponding integral converges. It is anyhow worth stressing that a quite complete theory has developed in [Ch84] for the case $V \equiv 0$ also considering noncompact manifolds neither spectral measures, but thinking the non compact manifold as a limit of compact (generally with boundaries) manifolds. In recent years, the theory of heat kernel and ζ function in symmetric manifolds has been developed on mathematical and physical grounds also proving the validity of (36) in noncompact symmetric manifolds [CH, BCVZ96].
There exist a quite large theoretical-physics literature on applications of these topics [BCVZ96] in quantum field theory in curved background, concerning particular cases and also higher

spin of the field (see [IM96, MI97] for the case of photons and gravitons spaces containing conical singularities), in particular, in the presence of noncompact manifolds containing conical singularities very important within QFT in the presence of a black hole [ZCV96]. (Problems related to the heat kernel and the ζ function in the case of a compact manifold containing conical singularities is not trivial on a mathematical point of view, it was treated by J. Cheeger in [Ch83].). It is known that, in the case $V \equiv m^2$ at least, both the regularity (including the positivity) of $K(t, x, y)$ and the heat kernel expansion (19) do not depend on the compactness of the manifold provided further hypotheses on $\mathcal{M}$ and A are given (see [Wa79, Ch84, Da89]). In particular, the behavior at $t \to 0^+$ is the same as in the compact case and one finds the asymptotic expansion (19) once again [Wa79]. Generally speaking, provided $K(t, x, y)$ is given through a spectral measure and A is positive definite, one can still prove, on any compact $K \subset \mathcal{M}$, (99) where now $\lambda = \inf \sigma(A)$. In this way $\zeta(s, x, y|A/\mu^2)$ can be defined and the results of **Theorem 2.2** remain substantially unchanged. General estimates on (x, y)-uniform bounds of $K(t, x, y|A)$ at large and little t can be found in [Ch84, Da89] for the pure Laplacian imposing further requirement on the geometry of the manifolds and bounds on the Ricci operator and sectional curvatures. The presence of vanishing (proper) eigenvalues can be treated similarly to the case of compact manifolds, subtracting the contribution of the corresponding eigenvectors from the heat kernel as given in (27). The existence of the integrated ζ function is much more difficult to study in the general case of a noncompact manifold, and, barring very particular situations (e.g. the Euclidean section of anti de Sitter spacetime), the integrated ζ function does not exist and one needs some volume cutoffs. Recently, other ways to overcome this shortcoming has been pointed out [Mü98]. The problem of the existence of the integrated ζ function is dramatically important in the issue of the computation of thermodynamical quantities of fields propagating in the spacetime around a black hole (and thus in the general issue of the black hole entropy). In that case also horizon divergences appear and their rôle and involved mathematics is not completely understood also because different integrated renormalization procedures disagree [ZCV96, IM96, MI97, Mo97b, FF98, Ie98]. Finally, the issue whether or not $-(2\beta)^{-1}\zeta'(0, x|A/\mu^2)$ defines the true local density of free energy still remains an open question (see the discussion in [Mo97a]). The presence of boundaries, maintaining the compactness, changes the results obtained in the case above ("closed" manifold) only for the presence of further terms in the heat-kernel expansion (19) depending on the boundaries [Ch84] and with noninteger powers of t. These terms can be interpreted as distributions concentrated on the boundary of the manifold (see [EORBZ94, El95] for the corresponding bibliography and physical applications). Obviously, in this case A' is not essentially self-adjoint and the considered self-adjoint extension depends on the imposed boundary conditions (Dirichlet/Neumann/Mixed problem) and the choice is related to the particular quantum state one is investigating.

2.3 $S[A]$ and $\mathcal{L}(x|A)$.

The results of these two subsections are quite known [Wa79, BD82, Fu91] in particular cases (e.g. $D = 4$, $V = m^2 + \xi R$, $m^2 > 0$) and, barring [Wa79], just in a formal way. Herein, we produce a rigorous proof of the substantial equivalence of the point-splitting approach and ζ

function procedure as far effective Lagrangian is concerned, starting from our hypotheses[5] in the general case $D > 1$. We shall give also some comments on some formal definitions often assumed by physicists. We remark also that, the relation between the local ζ function approach and the point-splitting procedure is now discussed in terms of the expansion (15) instead of the Schwinger-DeWitt one (hence, differently from other papers on the same subject, the obtained formulae do not distinguish between the cases of a finite or vanishing mass of the field) and a quite general scalar operator is considered here. In favour of Schwinger-DeWitt's expansion, it might be noticed anyhow that this expansion, at least formally and for m strictly positive, should work also in noncompact manifolds due to the sharp decay of the exponential $\exp(-m^2 t)$ in the heat kernel (see discussion in [Wa79]).

Definition 2.4. *Within the general hypotheses on $\mathcal{M}$ and A' and $r > 0$, the* **effective action** *associated to the operator A, is defined, within the ζ function approach, as*

$$S[A]_{\mu^2} := -\ln Z[A]_{\mu^2} \, , \tag{37}$$

where the **partition function** *$Z[A]_{\mu^2}$ in the right hand side is defined as*

$$Z[A]_{\mu^2} := \left[\det\left(\frac{A}{\mu^2}\right)\right]^{-1/2} = e^{\frac{1}{2}\frac{d}{ds}|_{s=0}\zeta(s|A/\mu^2)} \, . \tag{38}$$

Therefore we have defined $Z[A]$ by (4), the left hand side being rigorously interpreted as in (35) in the framework of the ζ-function. The definition of the effective Lagrangian is similar. The most natural choice is the following definition (where from now on the prime on a ζ function means the s derivative)

Definition 2.5. *Within the general hypotheses on $\mathcal{M}$ and A' and $r > 0$, the* **effective Lagrangian** *associated to the operator A, is defined, within the ζ function approach, by*

$$\mathcal{L}(x|A)_{\mu^2} := -\frac{1}{2}\zeta'(0, x|A/\mu^2) \, . \tag{39}$$

Notice that (5) is now fulfilled by definition of integrated ζ function (31). Furthermore, it is worth stressing that definition (39) is well-posed because **Theorem 2.2** states that no singularity can apper at $s = 0$ in the local ζ function.

We want to comment this definition to point out what such a definition actually "regularizes". This is also to make precise what is actually allowed and what is forbidden within the ζ function approach. Following [Wa79] and starting from (37), using (38) and (35), one has correctly (we omit the index μ^2 for sake of simplicity in the notations)

$$S[A] = -\ln Z[A] = -\ln\left[\det\left(\frac{A}{\mu^2}\right)\right]^{-1/2} \, . \tag{40}$$

[5]These hypotheses and our way are different from those used in [Wa79] which considered the case $V \equiv m^2$ only. There (also dropping the hypotheses of a compact manifold) the Schwinger-DeWitt expansion and an explicit use of large t behaviour of (t-derivatives of) $K(t, x, y)$ were used together with, in part, hypotheses of the essentially self-adjointness of A'^n for any $n \in \mathbb{N}$. The procedures used in the fundamental book [BD82] are very formal and no mathematical discussion appear.

At this step and quite often, physicists assume the validity of the matrix identity (for the moment we omit the factor μ^2 which is not necessary)

$$\mathrm{Tr} \ln A = \ln \det A \,. \tag{41}$$

One may define at this end (in the strong operatorial topology)

$$\ln A = \lim_{\epsilon \to 0^+} \left\{ \int_\epsilon^{+\infty} dt \, \frac{e^{-tA}}{t} + (\gamma - \ln \epsilon) I \right\} \,. \tag{42}$$

Anyhow, the trace of this operator *does not exist* at least because of the presence of γI. Also a direct definition of $\ln A$ by spectral theorem would prove that $\ln A$ is not a bounded operator and thus, *a fortiori*, it is not a trace-class operator. We conclude that (41) *makes no sense* in any cases, neither within the ζ-function approach. Anyhow, it is still possible to grasp our definition by this way employing a completely formal sequence of identities. The way is just to drop the annoying term in (42) as well as the regulator ϵ and write down through (41), using (17), (27) (dropping $P_0(x,x)$ for sake of simplicity) and interchanging several times the order of trace symbol and integrals and the s-derivative

$$\begin{aligned}
-2S[A] &= \mathrm{Tr} \ln A = \int_0^{+\infty} \frac{dt}{t} \, \mathrm{Tr}\, e^{-tA} = \int_0^{+\infty} \frac{dt}{t} \, \mathrm{Tr}\, K_t \\
&= \int_{\mathcal{M}} d\mu_g(x) \int_0^{+\infty} dt \, t^{-1} K(t,x,x|A) \\
&= \int_{\mathcal{M}} d\mu_g(x) \int_0^{+\infty} dt \, \frac{d}{ds}|_{s=0} \left(\frac{1}{\Gamma(s)} t^{s-1} \right) K(t,x,x|A) \\
&= \int_{\mathcal{M}} d\mu_g(x) \, \zeta'(0,x|A) \,.
\end{aligned} \tag{43}$$

Notice that, above, also the t-integration of the heat kernel times t^{-1} evaluated for $x = x$ trivially diverges, and thus also the first passages above are incorrect. Anyhow, looking at the last side, it is natural, from the formal identities above to get the definition (39). $\mathcal{L}(x|A)$ may be hence considered as the "formal" integral kernel of the operator $\ln A$ evaluated on the diagonal.

As a final comment on (41) we remark that, nevertheless, one could use this (literally wrong) identity to *define* an extension of the concept of the trace of $\ln A$, this is because the right hand side is however well defined. Anyhow, this way leads to a generally *not linear* trace due to the well-know *multiplicative anomaly* of the determinant defined in terms of ζ function [EVZ97].

2.4 Effective action and point-splitting procedure.

Let us consider the relation between the employed definitions and the point-splitting renormalization procedure. The idea of the point-splitting procedure [Wa79, BD82] consists, following the formal passages developed in (43), of the formal definition for the effective Lagrangian

$$\mathcal{L}(y|A) := \lim_{x \to y} \left[-\int_0^{+\infty} \frac{dt}{2t} \, K(t,x,y|A) - \text{"divergences"} \right] . \tag{44}$$

The divergences are those which appear evaluating the limit in the integral above by brute force [Wa79, BD82]. We shall find the precise form of these by our local ζ function approach. We notice that the term "divergences" is quite ambiguous, because a divergent term plus a finite term is always a divergent term. Such an ambiguity could be actually expected [Wa79] because of dependence on μ at least, which has to remain into the final expression of the renormalized effective action for several reasons [Wa79, Ha77, BD82] at least for $D = 4$. Therefore the final expression should contain a finite renormalization part dependent on the arbitrary scale μ. In practice, the actual value of μ can be fixed by experimental results (at least for $D = 4$). As a further general comment, which is not so often remarked, we stress that the point-splitting procedure *works only in the case* $P_0(x, y) \equiv 0$. Indeed, whenever $\lambda = 0$ is an eigenvalue of A, the integral for $x \neq y$ in (44) diverges due to the large t behaviour of the integrand. To avoid this drawback, one could try to change the integrand into $K(t, x, y|A) - P_0(x, y)$. Nevertheless this is not enough, indeed, a straightforward check at $t \to 0^+$ using **Theorem 1.3**, proves that the integrand so changed inserted in (44) produces a divergent integral at $t \to 0^+$ just because of the presence of P_0.

We need a definition and a useful lemma to prove that, within our general hypotheses and supposing also $r > 0$, the previous subtraction of divergences together with the coincidence limit are actually equivalent to the definition given in the local ζ function framework (39).

Definition 2.6. *Within our general hypotheses on* $\mathcal{M}$ *and* A' *and* $r > 0$*, for* N *integer* $> D/2+2$ *and* $\mu^2, \mu_0^2 > 0$ *fixed, the* N **truncated local** ζ **function** *is defined as the function of* $s \in \mathbb{C}$*,* $x, y \in \mathcal{M}$ *where the right hand side makes sense*

$$\begin{aligned}\zeta(N, s, x, y|A/\mu^2, \mu_0^{-2}) \quad &:= \quad \zeta(s, x, y|A/\mu^2) + \left(\frac{\mu}{\mu_0}\right)^{2s} \frac{P_0(x, y)}{s\Gamma(s)} \\ &-\frac{\mu^{2s}\chi(\sigma(x, y))}{(4\pi)^{D/2}\Gamma(s)} \sum_{j=0}^{N} a_j(x, y|A) \int_0^{\mu_0^{-2}} \frac{dt}{t} t^{s+j-D/2} e^{-\frac{\sigma(x,y)}{2t}} \end{aligned} \qquad (45)$$

Several properties of the function defined above are analized in the proof of **Theorem 2.2** given in **Appendix**.

Lemma 2.1. *Within our general hypotheses on* $\mathcal{M}$*,* A' *and for* $r > 0$*,* $\mu, \mu_0 > 0$ *and* $N > D/2 + 2$*, the function* $(s, x, y) \mapsto \zeta(N, s, x, y|A/\mu^2, \mu_0^2)$ *is analytic in* $\{s \in \mathbb{C} \mid Re\ s > D/2 - N\}$ *and belongs to* $C^0(\{s \in \mathbb{C} \mid Re\ s > D/2 - N\} \times \mathcal{M} \times \mathcal{M})$ *together with all of its* s *derivatives. Moreover*

$$\zeta'(N, 0, x, y|A/\mu^2, \mu_0^{-2}) = \Gamma(s)\zeta(N, s, x, y|A/\mu^2, \mu_0^{-2})|_{s=0} \qquad (46)$$

Finally,

$$\zeta(s, x|A/\mu^2) \quad = \quad \zeta(N, s, x, x|A/\mu^2, \mu_0^{-2}) - \frac{(\mu/\mu_0)^{2s} P_0(x, x)}{s\Gamma(s)}$$

$$
+\frac{\mu^{2s}}{(4\pi)^{D/2}}\sum_{j=0}^{N}\frac{a_j(x,x|A)(\mu_0^{-2})^{(s+j-D/2)}}{\Gamma(s)(s+j-D/2)}\,. \tag{47}
$$

Proof. The first part and the last identity are proven within the proof of **Theorem 2.2**. (46) is a trivial consequence of (101) in the proof of **Theorem 2.2** noticing that, there, the derivative at $s=0$ of the analytic continuation of $1/\Gamma(s)$ is equal to 1 and $1/\Gamma(s)\to 0$ as $s\to 0$. $\square$

Now, let us consider the identity (47) ($N > D/2+2$) in the case $P_0\equiv 0$ and evaluate the s derivative for $s=0$ necessary to get $\mathcal{L}(x|A)$ by (39). We have

$$
\begin{aligned}
\zeta'(0,y|A/\mu^2) &= \zeta'(N,0,y,y|A/\mu^2,\mu_0^{-2})-\sum_{j=0,j\neq D/2}^{N}\frac{\mu_0^{(D-2j)}a_j(y,y|A)}{(4\pi)^{D/2}(D/2-j)}\\
&\quad+\delta_D\left[\gamma+\ln\left(\frac{\mu}{\mu_0}\right)^2\right]\frac{a_{D/2}(y,y|A)}{(4\pi)^{D/2}}\,,
\end{aligned} \tag{48}
$$

where as usual, $\delta_D=1$ if D is even and $\delta_D=0$ if D is odd. From **Lemma 2.1**, we get

$$
\zeta'(N,0,y,y|A/\mu^2,\mu_0^{-2}) = \lim_{x\to y}\left[\Gamma(s)\zeta(N,s,x,y|A/\mu^2,\mu_0^{-2})|_{s=0}\right]\,.
$$

By (45), taking also account of $P_0\equiv 0$ in (27), the right hand side of the equation above reads

$$
\begin{aligned}
&\lim_{x\to y}\left[\Gamma(s)\zeta(s,x,y|A/\mu^2)|_{s=0}-\sum_{j=0}^{N}\frac{a_j(x,y|A)}{(4\pi)^{D/2}}\int_0^{\mu_0^{-2}}dt\,t^{j-D/2-1}e^{-\sigma(x,y)/2t}\right]\\
=\;&\lim_{x\to y}\left[\int_0^{+\infty}\frac{dt}{t}\,K(t,x,y|A)-\sum_{j=0}^{N}\frac{a_j(x,y|A)}{(4\pi)^{D/2}}\int_0^{\mu_0^{-2}}dt\,t^{j-D/2-1}e^{-\sigma(x,y)/2t}\right]
\end{aligned} \tag{49}
$$

Notice that the integral of the heat-kernel times t^{-1} converges away from the diagonal as follows from the proof of **Theorem 2.2** (see **Appendix**). To conclude, we have just to substitute the obtained result into the right hand side of (48) and using definition (39) and finally we have

$$
\begin{aligned}
\mathcal{L}(y|A) &= \lim_{x\to y}\left\{-\int_0^{+\infty}\frac{dt}{2t}\,K(t,x,y|A)+\sum_{j=0}^{N}\frac{(\frac{\sigma}{2})^{j-D/2}a_j(x,y|A)}{2(4\pi)^{D/2}}\int_{\frac{\sigma\mu_0^2}{2}}^{+\infty}du\,u^{D/2-j-1}e^{-u}\right.\\
&\quad\left.+\sum_{j=0,j\neq D/2}^{N}\frac{\mu_0^{(D-2j)}a_j(y,y|A)}{2(4\pi)^{D/2}(D/2-j)}-\delta_D\left[\gamma+\ln\left(\frac{\mu}{\mu_0}\right)^2\right]\frac{a_{D/2}(y,y|A)}{2(4\pi)^{D/2}}\right\}\,.
\end{aligned} \tag{50}
$$

It is possible to compute more explicitly the integrals above and give a close form of the divergent terms in (50). This can be done expanding the integrals in powers/logarithm of σ and keeping both the dominant divergent terms and those constant only. In the following $j=0,1,2,\ldots D/2+$

$1,\dots N$ ($N \geq D/2+2$) and $D > 1$. Let us define, with a little abuse of notation since the right hand side is not function of σ only,

$$H_j(\sigma) := \frac{(\frac{\sigma}{2})^{j-D/2} a_j(x,y|A)}{2(4\pi)^{D/2}} \int_{\frac{\sigma\mu_0^2}{2}}^{+\infty} du\, u^{D/2-j-1} e^{-u}\,. \tag{51}$$

Therefore one gets by some computations, $O_k(\sigma)$ being functions which vanish as $x \to y$, in the case of D even

$$\begin{aligned}
H_{j\geq D/2+1}(\sigma) &= -\frac{\mu_0^{(D-2j)}}{D/2-j}\frac{a_j(y,y|A)}{2(4\pi)^{D/2}} + O_j(\sigma)\,; &(52)\\
H_{D/2}(\sigma) &= -\frac{a_{D/2}(x,y|A)}{2(4\pi)^{D/2}}\ln\left(\frac{\sigma\mu_0^2}{2}\right) - \gamma\frac{a_{D/2}(y,y|A)}{2(4\pi)^{D/2}} + O_{D/2}(\sigma)\,; &(53)\\
H_{j<D/2}(\sigma) &= (D/2-j-1)!\frac{a_j(x,y|A)}{2(4\pi)^{D/2}}\left(\frac{2}{\sigma}\right)^{D/2-j} - \frac{\mu_0^{(D-2j)}}{D/2-j}\frac{a_j(y,y|A)}{2(4\pi)^{D/2}} + O_j(\sigma) &(54)
\end{aligned}$$

and, whenever D is odd,

$$\begin{aligned}
H_{j>(D+1)/2}(\sigma) &= -\frac{\mu_0^{(D-2j)}}{D/2-j}\frac{a_j(y,y|A)}{2(4\pi)^{D/2}} + O_j(\sigma)\,; &(55)\\
H_{(D+1)/2}(\sigma) &= \frac{a_{(D+1)/2}(y,y|A)}{\mu_0(4\pi)^{D/2}} + O_{(D+1)/2}(\sigma)\,; &(56)\\
H_{(D-1)/2}(\sigma) &= \frac{a_{(D-1)/2}(x,y|A)}{2(4\pi)^{D/2}}\sqrt{\frac{2\pi}{\sigma}} - \frac{\mu_0 a_{(D-1)/2}(y,y|A)}{(4\pi)^{D/2}} + O_{(D-1)/2}(\sigma)\,; &(57)\\
H_{j<(D-1)/2}(\sigma) &= \left(\frac{2}{\sigma}\right)^{D/2-j}\frac{(D-2-2j)!!\sqrt{\pi}}{2^{(D+1)/2-j}}\frac{a_j(x,y|A)}{(4\pi)^{D/2}} - \frac{\mu_0^{(D-2j)}}{D/2-j}\frac{a_j(y,y|A)}{2(4\pi)^{D/2}}\\
&\quad +O_j(\sigma)\,. &(58)
\end{aligned}$$

By substituting the results above into (50), we have

Theorem 2.3 *Within our general hypotheses on $\mathcal{M}$ and A', for $r > 0$ and $\mu > 0$ and $D > 1$, the effective action computed by* (39) *can be also computed by a point-splitting procedure provided $P_0 \equiv 0$. Indeed, whenever D is even*

$$\begin{aligned}
\mathcal{L}(y|A)_{\mu^2} &= \lim_{x\to y}\left\{-\int_0^{+\infty}\frac{dt}{2t}\,K(t,x,y|A) - \frac{a_{D/2}(x,y|A)}{2(4\pi)^{D/2}}\ln\left(\frac{\sigma\mu^2}{2}\right)\right.\\
&\quad\left. + \sum_{j=0}^{D/2-1}(D/2-j-1)!\frac{a_j(x,y|A)}{2(4\pi)^{D/2}}\left(\frac{2}{\sigma}\right)^{D/2-j}\right\} - 2\gamma\frac{a_{D/2}(y,y|A)}{2(4\pi)^{D/2}}\,, &(59)
\end{aligned}$$

and, whenever D is odd.

$$\mathcal{L}(y|A)_{\mu^2} = \lim_{x\to y}\left\{-\int_0^{+\infty}\frac{dt}{2t}\,K(t,x,y|A) + \sqrt{\frac{2\pi}{\sigma}}\frac{a_{(D-1)/2}(x,y|A)}{2(4\pi)^{D/2}}\right.$$

$$+\sum_{j=0}^{(D-3)/2} \frac{(D-2j-2)!!\sqrt{\pi}}{2^{(D+1)/2-j}} \frac{a_j(x,y|A)}{(4\pi)^{D/2}} \left(\frac{2}{\sigma}\right)^{D/2-j} \Bigg\} . \tag{60}$$

Comments.

(1) When D is odd, μ disappears from the final results. Conversely, when D is *even*, the scale μ appears, and this is necessary due to the logarithmic divergence in (59), indeed, it has to combine with σ in order to give a nondimensional argument of the logarithm. Since the presence of μ in (59), the left hand side is ambiguously defined because it can be changed by adding terms of the form, where α is any strictly positive real,

$$\delta\mathcal{L}(x,\alpha|A) := -\delta_D \frac{a_{D/2}(x,x|A)}{2(4\pi)^{D/2}} \ln\alpha \,, \tag{61}$$

This correspond trivially to a rescaling of μ^2: $\mu^2 \to \alpha\mu^2$. These terms cannot be determined within this theory and represent a remaining finite part of the renormalization procedure. The pointed out ambiguity is a subcase of an ambiguity which arises also in the presence of P_0. This can be carried out directly from (39). In fact, directly from the definitions (30) and (27), we have $\zeta(s,x|A/(\alpha\mu^2)) = \alpha^s\zeta(s,x|A/\mu^2)$ and thus $\zeta'(0,x|A/(\alpha\mu^2)) = \zeta'(0,x|A/\mu^2) + \zeta(0,x|A/\mu^2)\ln\alpha$. Reminding (34) and (39) we get

$$\mathcal{L}(x|A)_{\alpha\mu^2} = \mathcal{L}(x|A)_{\mu^2} - \left[\delta_D \frac{a_{D/2}(x,x|A)}{2(4\pi)^{D/2}} - \frac{P_0(x,x)}{2}\right] \ln\alpha \,. \tag{62}$$

(2) All these results should remain unchanged also in the case of a noncompact manifold because all proofs was based on **Theorem 2.2**, which, as discussed in **2.2** should hold true also dropping the hypothesis of compactness (and assuming some further hypotheses as completeness).

2.5 $\langle\phi^2(x|A)\rangle$ and local ζ function.

Let us consider the local ζ-function definition of the field fluctuations $\langle\phi^2(x|A)\rangle$ [IM98]. The main definitions [IM98] are the following ones

Definition 2.7. *Within our general hypotheses on $\mathcal{M}$ and A' and $r > 0$, the* **field fluctuation** *of the field associated to the operator A are defined by*

$$\langle\phi^2(x|A)\rangle_{\mu^2} := \frac{d}{ds}|_{s=0} Z(s,x|A/\mu^2) \,, \tag{63}$$

where the **local ζ function of the field fluctuations** *$Z(s,x|A/\mu^2)$ is defined as the function of x and s whenever the right hand side is sensible, for any $\mu^2 > 0$*

$$Z(s,x|A/\mu^2) := \frac{s}{\mu^2}\zeta(s+1,x|A/\mu^2) \,. \tag{64}$$

Concerning the mathematical consistency of the proposed definitions, from **Theorem 2.2**, we have

Theorem 2.4. *In our hypotheses on $\mathcal{M}$ and A' and $r > 0$, for $\mu^2 > 0$, $(s, x) \mapsto Z(s, x|A/\mu^2)$ is a meromorphic function of s analytic in $s = 0$ and the only possible poles are* simple *poles and are situated in the points*

$$\begin{aligned} s_j &= D/2 - j - 1, \quad j = 0, 1, 2, \dots \qquad \textit{if } D \textit{ is odd, or} \\ s_j &= D/2 - j - 1, \quad j = 0, 1, 2, \dots D/2 - 2 \quad \textit{if } D \textit{ is even.} \end{aligned}$$

Moreover the function Z and all of its s derivatives belong to $C^0((\mathbb{C} - \mathcal{P}) \times \mathcal{M})$, $\mathcal{P}$ being the set of the actual poles (each for some values of x) among the points listed above.

Comments.

(1) Let us summarize the formal procedure which leads one to the definitions above [IM98]. The general idea consists of considering the following *purely formal* identity which takes account of the formal definition (6) and the rigorous identity (31)

$$\begin{aligned} \langle \phi^2(x|A) \rangle_{\mu^2} &= -\frac{\delta}{\delta J(x)}|_{J\equiv 0} \frac{1}{2}\frac{d}{ds}|_{s=0} \sum_{j\in\mathbb{N}}{}' \left\{ \frac{\lambda_j[A+2J]}{\mu^2} \right\}^{-s} \\ &= -\frac{d}{ds}|_{s=0} \sum_{j\in\mathbb{N}}{}' \frac{\delta}{\delta J(x)}|_{J\equiv 0} \left\{ \frac{\lambda_j[A+J]}{\mu^2} \right\}^{-s} . \end{aligned} \tag{65}$$

Then one formally computes the functional derivatives (Gâteaux derivatives) of $\{\lambda_j[A+J]/\mu^2\}^{-s}$ at $J \equiv 0$ [IM98, Mo97a] obtaining

$$\frac{\delta}{\delta J(x)}|_{J\equiv 0} \left\{ \frac{\lambda_j[A+J]}{\mu^2} \right\}^{-s} = -\frac{s}{\mu^2} \left[\frac{\lambda_j}{\mu^2} \right]^{-(s+1)} \phi_j(x)\phi_j^*(x)\sqrt{g(x)}. \tag{66}$$

This result, inserted in (65) and interpreting the final series in the sense of the analytic continuation, gives both (63) and and (64). Obviously one could try to give some rigorous meaning to the formal passages above, but this is not our approach, which assumes (63) and (64) by *definition.*

(2) In the case $\zeta(s, x|A/\mu^2)$ has no pole at $s = 1$, namely, when D is odd or when D is even and $a_{D/2-1}(x, x) = 0$ (see **Theorem 2.2**) (63) reduces to the trivial formula, which does not depend of the value of μ^2 (this follows directly from the definition of local ζ function (30))

$$\langle \phi^2(x|A) \rangle_{\mu^2} = \mu^{-2}\zeta(1, x|A/\mu^2) = \zeta(1, x|A) . \tag{67}$$

(3) We finally remark that, in [IM98], definitions (63) and (64) have been checked on several concrete cases obtaining a perfect agreement with other renormalization procedures, also concerning the remaining finite renormalization part related to the μ^2 ambiguity. The local ζ-function approach concerning the field fluctuations has produced also a few new results, e.g., the general form for renormalized trace of the one-loop stress tensor in the generally nonanomalous case, and several applications in symmetric spaces for general values of the parameter ξ which fixes the coupling of the field with the curvature (see [IM98]).

2.6 $\mu^{-2n}\zeta(n,x,y|A/\mu^2)$ as Green function of A^n.

Let us consider the usual operator A, the Friedrichs extension of A' given in (1). Let us also suppose explicitly that $P_0 \equiv 0$ namely, $\sigma(A) \subset (0,+\infty)$. In this case A has a well-defined unique inverse operator $A^{-1} : R(A) \to \mathcal{D}(A)$. We notice that A^{-1} is bounded by $\sup\{1/\lambda|\lambda \in \sigma(A)\} < +\infty$. Moreover $R(A)$ is dense in $L^2(\mathcal{M}, d\mu_g)$, because $R(A) = R(A-0I)$ which is dense A being self-adjoint and 0 belonging to the resolvent $\rho(A)$. Therefore A^{-1} can be uniquely extended into a bounded operator defined on the whole $L^2(\mathcal{M}, d\mu_g)$ which we shall indicate with the same symbol A^{-1}. (Alternatively, one can check on the fact that $R(A)$ is dense in $L^2(\mathcal{M}, d\mu_g)$ directly from the spectral representation of A, where the series is understood in the strong topology,

$$A = \sum_{j=0}^{+\infty} \lambda_j \phi_j (\phi_j, \quad) \, , \tag{68}$$

taking account that the vectors ϕ_j defines a Hilbertian base of $L^2(\mathcal{M}, d\mu_g)$). By definition of inverse operator, $AA^{-1} = I$ holds true in the whole space and not only in $R(A)$, being $A^{-1}(L^2(\mathcal{M}, d\mu_g)) \subset \mathcal{D}(A)$ (this follows from the fact that $A = A^\dagger$ is a closed operator and A^{-1} is bounded in the dense set $R(A)$). The other relation $A^{-1}A = I_{\mathcal{D}(A)}$ holds true in $\mathcal{D}(A)$ as indicated by the employed notation.
We are now interested in integral representations of A^{-1}. In the case $D < 4$, one gets from **Theorem 1.4** that the series of elements $||A^{-1}\phi_j||^2 = |\lambda_j|^{-2} = \lambda_j^{-2}$ converges and, since $\{\phi_j | j = 0, 1, 2, \ldots\}$ is a Hilbertian base, this means that [RS] A^{-1} is a Hilbert-Schmidt operator and thus, holding our hypotheses of a countable topology involving a consequent separable measure, A^{-1} is represented by an $L^2(\mathcal{M} \times \mathcal{M})$ integral kernel $A^{-1}(x, y)$. As is well known, such a function is called a *Green function* of A and satisfies almost everywhere, for any $\psi \in \mathcal{D}(A)$,

$$\int_{\mathcal{M}} d\mu_g(y)\, A^{-1}(x,y)(A\psi)(y) = \psi(x) \qquad (\text{ namely } A^{-1}A = I_{\mathcal{D}(A)}) \tag{69}$$

and (almost everywhere), for any $\psi \in L^2(\mathcal{M}, d\mu_g)$

$$A_x \int_{\mathcal{M}} d\mu_g(y)\, A^{-1}(x,y)\psi(y) = \psi(x) \qquad (\text{ namely } AA^{-1} = I) \tag{70}$$

These relationships are often written in the synthetic form $A_x A^{-1}(x,y) = \delta(x,y)$
In the case $D \geq 4$, A^{-1} is not Hilbert-Schmidt since the series of elements $|\lambda_j|^{-2} = \lambda_j^{-2}$ diverges as follows from **Theorem 1.4**. Anyhow, these facts do not forbid the existence of integral kernels, which are *not* $L^2(\mathcal{M} \times \mathcal{M})$, which represent A^{-1} and satisfy either (69) and/or (70). In any cases, for $P_0 \equiv 0$, we can state the following simple result.

Proposition 2.1 *Within our general hypotheses on $\mathcal{M}$ and A', supposing also $P_0 \equiv 0$,* if *a locally y-integrable $B(x,y)$ exits which satisfy either* (69) *or* (70) *with $B(x,y)$ in place of $A^{-1}(x,y)$, it must be unique (barring differences on vanishing measures sets of $\mathcal{M} \times \mathcal{M}$) and the operator B defined by this integral kernel must coincide with A^{-1}. Furthermore, respectively,* (70) *or* (69)

has to hold true with $B(x,y)$ in place of $A^{-1}(x,y)$.

Proof. The uniqueness is a trivial consequence of the linearity of (69) or (70) respectively, taking account that $R(A)$ is dense in $L^2(\mathcal{M}, d\mu_g)$ in the first case, and that A is injective in the other case. The coincidence $B = A^{-1}$ follows by pure algebraic considerations straightforwardly. Then, the last point of the thesis follows from the definition of inverse operator. □

Comments.
(1) Dropping the hypotheses of $P_0 \equiv 0$, A^{-1} does not exist and also any integral kernel $B(x,y)$ which satisfies (70) on $R(A)$, if exists, cannot be uniquely determined since $B(x,y) + c\phi(x)\phi^*(y)$ satisfies the same equation for any $c \in \mathbb{C}$ whenever $A\phi = 0$. (Notice that ϕ can be redefined on a set of vanishing measure so that it belongs to $C^\infty(\mathcal{M})$ by **Theorem 1.1**).
(2) We stress that, **Proposition 2.1** is much more general, indeed, trivially, it holds true also considering $A : \mathcal{D}(A) \to \mathcal{H}$, where $\mathcal{D}(A)$ is a dense subspace (not necessarily closed) of a Hilbert space $\mathcal{H} = L^2(X, d\mu)$, μ being any positive measure on X, $A = A^\dagger$ and $\sigma(A) \subset (0, +\infty)$.
(3) In particular, for $Ker\ A = \{0\}$, **Proposition 2.1** holds true considering A^n, for any positive integer n, rather than A self, where A is the Friedrichs extension of A' and A^n is defined via spectral theorem as usual.
(4) Notice that, in this case A^{-n} is trace class if and only if $n > D/2$ as consequence of **Theorem 1.4**.

The following Theorem gives a practical realization of the Green functions of A^n in terms of $\mu^{-2n}\zeta(n, x, y|A/\mu^2)$.

Theorem 2.5. *Within our general hypotheses on A' and $\mathcal{M}$, supposing also $r > 0$ and $P_0 \equiv 0$ and fixing any positive integer n,*

(a) *$(x,y) \mapsto \mu^{-2n}\zeta(n, x, y|A/\mu^2)$, not depending on $\mu^2 > 0$, for $(x,y) \in (\mathcal{M} \times \mathcal{M}) - \mathcal{D}$ ($\mathcal{D} = \{(x,y) \in \mathcal{M} \times \mathcal{M} \mid x = y\}$) defines the unique Green function of A^n,*

$$G(x,y|A^n) := \zeta(n, x, y|A) \tag{71}$$

in the sense that this is the unique $C^0((\mathcal{M}\times\mathcal{M}) - \mathcal{D})$ function which satisfies for any $\psi \in \mathcal{D}(A^n)$ ($\mathcal{D}(A^n)$ is the domain of A^n obtained from the spectral theorem)

$$\int_{\mathcal{M}} d\mu_g(y)\, G(x,y|A^n)(A^n\psi)(y) = \psi(x) \text{ (almost everywhere)} \tag{72}$$

and, for any $\psi \in L^2(\mathcal{M}, d\mu_g)$

$$A^n_x \int_{\mathcal{M}} d\mu_g(y)\, G(x,y|A^n)\psi(y) = \psi(x) \text{ (almost everywhere)} \tag{73}$$

and thus defines the integral kernel of A^{-n}.

(b) *Dropping the hypothesis $P_0 \equiv 0$, $(x,y) \mapsto \mu^{-2n}\zeta(n, x, y|A/\mu^2) =: G(x,y|A^n)$ does not depend on $\mu^2 > 0$, and it satisfies* (72) *anymore for $\psi \in \mathcal{D}(A^n) \cap \{KerA\}^\perp$ and produces a vanishing right hand side for $\psi \in Ker\ A$; and it satisfies also* (73) *for any $\psi \in R(A^n)$.*

(c) *For $n > D/2$ $(D > 1)$, in both cases (a) and (b), $G(x, y|A^n)$ does not diverge for $y \to x$ and is continuous in $\mathcal{M} \times \mathcal{M}$. Moreover,*

$$\int_{\mathcal{M}} G(x, x|A^n) d\mu_g(x) = TrA^{-n} , \tag{74}$$

(where A^{-n} in the the trace is defined via spectral theorem dropping the part of the spectral measure on the kernel of A whenever it is not trivial).

Sketch of Proof. (See also [Ag91] for the item (c)) The uniqueness of the Green function in the case (a) is a trivial consequence of **Proposition 2.1** and the remark (3) above. The divergences as $x \to y$ in the Green functions of A^n given in (71) can be analyzed employing the truncated local ζ function as we done studying the effective Lagrangian. The remaining part of the theorem is based on the following identities used recursively, for any $\psi \in \mathcal{D}(A)$, where $'$ indicate the t derivative

$$\begin{aligned}
&\int_{\mathcal{M}} d\mu_g(y)\, \zeta(1, x, y|A)(A\psi)(y) = \int_0^{+\infty} dt \int d\mu_g(y)\, K(t, x, y)(A\psi)(y) \\
= \ & \lim_{\epsilon \to 0^+} \int_\epsilon^{1/\epsilon} \int d\mu_g(y)\, K(t, x, y)(A\psi)(y) = \lim_{\epsilon \to 0^+} \int_\epsilon^{1/\epsilon} dt \int d\mu_g(y)\, (A_y K(t, x, y))\psi(y) \\
= \ & - \lim_{\epsilon \to 0^+} \int_\epsilon^{1/\epsilon} dt \int d\mu_g(y)\, K(t, x, y)'\psi(y) = - \lim_{\epsilon \to 0^+} \int_\epsilon^{1/\epsilon} dt \left(\int d\mu_g(y)\, K(t, x, y)\psi(y) \right)' \\
= \ & - \lim_{\epsilon \to 0^+} (e^{-(1/\epsilon)A}\psi)(x) + \lim_{\epsilon \to 0^+} (e^{-\epsilon A}\psi)(x) = \psi(x) . \qquad \Box
\end{aligned}$$

We remark that, If P_0 is untrivial the Green function of A^n is not clearly defined because A^n is not injective, in fact, the "Green function" $G(x, y|A^n)$ defined via local ζ function (71) correspond to a possible choice for a right-inverse of the operator A^n. Conversely, for $P_0 \equiv 0$, $\zeta(n, x, y|A)$ is the unique Green function of A^n, and not A'^n. Actually in our case there is no ambiguity since A' determines unambiguously its self-adjoint extension A. However, all results given within this subsection hold true also considering manifolds which are compact with boundary. In that case it is possible to have different self-adjoint extensions of A' determined by different boundary conditions one may impose on the functions in $\mathcal{D}(A')$. In such a situation, provided $P_0 \equiv 0$, the Green function is still uniquely determined by the chosen self-adjoint extension of A' or, equivalently, by the chosen boundary conditions.

2.7 $\langle \phi^2(x|A) \rangle$, point-splitting and Hadamard expansion.

The procedure of the point-splitting for the field fluctuation is based, once again, upon a divergence subtraction procedure in the limit coincidence of the arguments of the Green function

$$\langle \phi^2(y|A) \rangle_{\mu^2} = \lim_{x \to y} \{G'(x, y|A) - \text{"divergences"} \} , \tag{75}$$

where $G'(x, y|A)$ is a "Green function" of the operator A, namely an integral kernel of the operator A^{-1} provided it exists. On the physical ground $G'(x, y|A)$ should determine the quantum

state completely after the "Lorentzian"-time analytic continuation [Ha77, Wa79, BD82, FR87, Fu91, Wa94] by determining the Feynman propagator as well as the Wightman functions of any order for the quasifree state [KW91].

Concerning the "divergences" above, we want to determine them directly from the ζ function approach assuming (71) in the case $n=1$ as the Green function to put in the expression above. Notice that this identification is automatic whenever $P_0 \equiv 0$ because of the uniqueness of the Green function proven above. Anyhow, we shall assume (71), for $n=1$, *also* in the case $Ker\, A \neq \{0\}$ where the concept of Green function is not so clearly understood.

Let us proceed as in the case of the effective action. We have, from (47),

$$\begin{aligned}
Z(s,y|A/\mu^2) &= \frac{s}{\mu^2}\zeta(s+1,y|A/\mu^2) \\
&= \frac{s}{\mu^2}\zeta(N,s+1,y,y|A/\mu^2,\mu_0^{-2}) - \left(\frac{\mu}{\mu_0}\right)^{2s+2}\frac{sP_0(y,y)}{\mu^2(s+1)\Gamma(s+1)} \\
&\quad + \frac{s\mu^{2s}}{(4\pi)^{D/2}}\sum_{j=0}^{N}\frac{a_j(y,y|A)(\mu_0^{-2})^{(s+1+j-D/2)}}{\Gamma(s+1)(s+1+j-D/2)}\,.
\end{aligned} \tag{76}$$

Using the definition (63) we have

$$\begin{aligned}
\langle\phi^2(x|A)\rangle &= \frac{1}{\mu^2}\zeta(N,1,y,y|A/\mu^2,\mu_0^{-2}) - \frac{P_0(y,y)}{\mu_0^2} + \delta_D\frac{a_{D/2-1}(y,y)}{(4\pi)^{D/2}}\left[\gamma + \ln\left(\frac{\mu}{\mu_0}\right)^2\right] \\
&\quad + \frac{1}{(4\pi)^{D/2}}\sum_{j=0;j\neq D/2-1}^{N}\frac{a_j(y,y|A)(\mu_0^{-2})^{(1+j-D/2)}}{(1+j-D/2)}\,,
\end{aligned} \tag{77}$$

where $\delta_D = 0$ if D is odd and $\delta_D = 1$ otherwise ($D>1$ in both cases). Now, we notice that, from **Lemma 2.1**, $\zeta(N,1,x,y|A/\mu^2,\mu_0^2)$ is continuous for $x\to y$ since it holds $N > D/2+2$. We can re-write $\zeta(N,1,x,y|A/\mu^2,\mu_0^2)$ in the right hand side of (77) by employing (45)

$$\begin{aligned}
\zeta(N,1,y,y|A/\mu^2) &= \lim_{x\to y}\left\{\zeta(1,x,y|A/\mu^2,\mu_0^{-2}) + \left(\frac{\mu}{\mu_0}\right)^2 P_0(x,y)\right. \\
&\quad \left. - \frac{\mu^2}{(4\pi)^{D/2}}\sum_{j=0}^{N}a_j(x,y|A)\int_0^{\mu_0^{-2}} dt\, t^{j-D/2}e^{-\sigma(x,y)/2t}\right\}.
\end{aligned}$$

Inserting this result in (77), we get

$$\begin{aligned}
\langle\phi^2(y|A)\rangle &= \delta_D\frac{a_{D/2-1}(y,y|A)}{(4\pi)^{D/2}}\left[\gamma + \ln\left(\frac{\mu}{\mu_0}\right)^2\right] \\
&+ \lim_{x\to y}\left\{\zeta(1,x,y|A) + \sum_{j=0;j\neq D/2-1}^{N}\frac{a_j(x,y|A)}{(4\pi)^{D/2}}\left[\frac{\mu_0^{-2(j-D/2+1)}}{j-D/2+1}\right.\right. \\
&- \left.\left(\frac{\sigma}{2}\right)^{j-D/2+1}\int_{\sigma\mu_0^2/2}^{+\infty} du\, u^{D/2-j-2}e^{-u}\right]
\end{aligned}$$

$$- \quad \delta_D \frac{a_{D/2-1}(x,y|A)}{(4\pi)^{D/2}} \int_{\sigma\mu_0^2/2}^{+\infty} du\, \frac{e^{-u}}{u} \Bigg\} \,. \tag{78}$$

Above, N is a fixed integer and $N \geq D/2+2$. The integrals above have been still computed in (52) – (58) since

$$\frac{a_{j+1}(x,y|A)}{2(4\pi)^{D/2}} \left(\frac{\sigma}{2}\right)^{j-D/2+1} \int_{\sigma\mu_0^2/2}^{+\infty} du\, u^{D/2-j-2} e^{-u} = H_{j+1}(\sigma) \,. \tag{79}$$

Using (52) – (58) in (78), we finally get the following theorem.

Theorem 2.6. *Within our general hypotheses on $\mathcal{M}$ and A', for $r > 0$ and $D > 1$ the field fluctuation computed by* (63) *can be computed also by a point-splitting procedure for $\langle \phi^2(y|A) \rangle$. Indeed,*

$$\begin{aligned} \langle \phi^2(y|A) \rangle_{\mu^2} \quad &= \quad \frac{2\gamma a_{D/2-1}(y,y|A)}{(4\pi)^{D/2}} \\ &+ \lim_{x\to y} \Bigg\{ G(x,y|A) - \sum_{j=0}^{D/2-2} (D/2-j-2)! \left(\frac{2}{\sigma}\right)^{D/2-j-1} \frac{a_j(x,y|A)}{(4\pi)^{D/2}} \\ &+ \frac{a_{D/2-1}(x,y|A)}{(4\pi)^{D/2}} \ln\left(\frac{\sigma\mu^2}{2}\right) \Bigg\} \,, \end{aligned} \tag{80}$$

whenever D is even. The term containing the sum over j appears for $D \geq 4$ only.

$$\begin{aligned} \langle \phi^2(y|A) \rangle_{\mu^2} \quad &= \quad \lim_{x\to y} \Bigg\{ G(x,y|A) - \sum_{j=0}^{(D-5)/2} \frac{(D-2j-4)!!\sqrt{\pi}}{2^{(D-3)/2-j}} \left(\frac{2}{\sigma}\right)^{D/2-j-1} \frac{a_j(x,y|A)}{(4\pi)^{D/2}} \\ &- \frac{a_{(D-3)/2}(x,y|A)}{(4\pi)^{D/2}} \sqrt{\frac{2\pi}{\sigma}} \Bigg\} \,, \end{aligned} \tag{81}$$

whenever D is odd. The term containing the sum over j appears for $D \geq 5$ only.

Comments.
(1) First of all, notice that μ^2 has disappeared from the final result in the case D is odd. Once again, the only task of μ^2 is to make physically sensible the argument of the logarithm in the case D is even.
(2) Eq. (80) and (81) prove that $G(x,y|A)$ has the Hadamard singular behaviour [Ga64] for $x \sim y$. Indeed, the terms after $G(x,y|A)$ in the right hand sides of the equations above, taking account that $\langle \phi^2(y|A) \rangle = Z'(0,y|A)$ is a regular function of x due to **Theorem 2.4**, give the singular part of $G(x,y|A)$ which is just that considered in building up perturbative Hadamard's local fundamental solutions [Ga64]. On the physical ground, this means that the quantum state associated to the Green function is Hadamard at least in the Euclidean section of the manifold. This is a very important point concerning the stress tensor renormalization procedure

[KW91, Wa94] by the point-splitting approach. Generally speaking, the *point splitting procedure* as known from the literature (see [BD82, Wa79, Fu91, Wa94] and references therein) consists of subtracting, from the Green function, a Hadamard local solution, namely, a C^∞ function of x defined in a normal convex neighborhood of y of the form

$$H_{LMN}(x,y) = \Theta_D \frac{U_L(x,y)}{(4\pi)^{D/2}(\sigma/2)^{D/2-1}} + \delta_D V_M(x,y)\ln(\sigma/2) + \delta_D W_N(x,y) \tag{82}$$

Where $U_L(x,y) = \sum_{j=0}^{L} u_j(x,y)\sigma^j$, $V_M(x,y) = \sum_{j=0}^{M} v_j(x,y)\sigma^j$ and $W_N(x,y) = \sum_{j=0}^{N} w_j(x,y)\sigma^j$, δ_D was defined previously. Moreover, $\Theta_D := 1$ for $D \neq 2$ and $\Theta_2 := 0$. All sums above are truncated to some orders L, M, N, in particular $L = D/2-2$ when D is even. The corresponding series: $M, N \to \infty$ for D even and in $L \to \infty$ for D odd, generally diverges. Actually, concerning our procedure, it is sufficient taking account of the divergent and finite terms for $x \to y$ in the formal series above.
There are recursive differential equations, obtained by considering the formal equation $A_x H = 0$ where $H = H_{\infty,\infty,\infty}$, that determine each term of the expansions above. In particular, the coefficients of the expansions of U and V are completely determined by requiring that $H_{\infty,\infty\infty}$, formally, is a Green function of A. This means that one has to fix opportunely the value of the coefficient of the leading divergence as $x \to y$ as said in the end of Section 2 in Chapter 5 of [Ga64]. In the practice, with our definition of the Riemannian measure, it must be $u_0(y,y) = 4\pi^{D/2}/[D(D-2)\omega_D]$ where ω_D is the volume of the unitary D-dimensional disk for $D \geq 3$, and $v_0(y,y) = 1/(4\pi)$ for $D = 2$. Similarly, the coefficients of the formal series for W_N are determined, for $j > 0$ only, once w_0 has been fixed. $w_0(x,y)$ self can be fixed arbitrarily.
We have proven by **Theorem 2.6** that the local ζ function procedure makes the same job made by the point splitting procedure. In particular, by a direct comparison between equations (23) (24) and the equations for u_i, v_j, w_j given in Chapter 5 Section 2 of [Ga64], and by a comparison between $u_0(x,y)$ and the corresponding terms in (80) and (81), one can check straightforwardly that the procedure pointed out in (80) and (81) consist of the coincidence limit of the difference between the Green function and Hadamard solutions with, respectively, $L = D/2-2$, $M = 0$, $N = 0$ for any even $D > 1$ and $L = D/2 - 3/2$ for any odd $D > 1$. Moreover, whenever D is even, the Hadamard solution is completely determined by choosing

$$w_0(x,y) = -\frac{a_{D/2-1}(x,y|A)}{(4\pi)^{D/2}}(2\gamma + \ln\mu^2) \tag{83}$$

Therefore, the local ζ function procedure picks out particular Hadamard solutions by a particular choice of $w_0(x,y)$. We stress that, within the point splitting procedure for the field fluctuations, it seems that there is no general way to choose a particular function $w_0(x,y)$ rather than another one (the situation is a bit different concerning the *stress tensor* where one can impose other constraints as the conservation of the renormalized stress tensor). Obviously there is no guarantee that the choice of $w_0(x,y)$ performed by the ζ function procedure is the physical one (if it exists).
Finally, we remark that the obtained result should hold also in the case $\mathcal{M}$ is not compact. In

this case, in general, A' admits different self-adjoint extensions and thus Green functions, corresponding to different physical states. The important point is that the Hadamard expansion does not depend on the considered self-adjoint extension. In other words, the singularity eliminated by the point-splitting procedure from the Green function is universal, depending on the local geometry only.
(3) In the general case, similarly to the results found for the effective Lagrangian, an ambiguity appears because the presence of the scale μ (for D even). In fact, any rescaling $\mu^2 \to \alpha\mu^2$ changes the value of $\langle\phi^2(x|A)\rangle_{\mu^2}$ producing

$$\langle\phi^2(x|A)\rangle_{\alpha\mu^2} = \langle\phi^2(x|A)\rangle_{\mu^2} + \delta_D \frac{a_{D/2-1}(x,x|A)}{(4\pi)^{D/2}} \ln\alpha \,. \tag{84}$$

It is worth stressing that this ambiguity concerns just the term $w_0(x,y)$ of the Hadamard expansion (see point (2) above). Therefore, in a general approach containing both local ζ function regularization and point-splitting procedure, the ambiguity pointed out above can be generalized into

$$\langle\phi^2(x|A)\rangle_{\alpha\mu^2,\delta w_0} = \langle\phi^2(x|A)\rangle_{\mu^2} + \delta_D \frac{a_{D/2-1}(x,x|A)}{(4\pi)^{D/2}} \ln\alpha - \delta_D \delta w_0(x,x) \,, \tag{85}$$

$\delta w_0(x,y)$ being any smooth function in $\mathcal{M}\times\mathcal{M}$ which could be fixed by imposing some further physical constraints.

2.8 Further properties of $Z(s,x|A/\mu^2)$ and $\langle\phi^2(x|A)\rangle$.

Within this section we want to prove a local version of a formula related to the field fluctuation and to the change of the mass in the field operator. Concerning the ζ function, similar formulae have appeared in [Ca90] with the hypothesis that $\mathcal{M}$ is a homogeneous space, and in [BCVZ96] for the integrated ζ function without a rigorous proof. Concerning the field fluctuations, similar formulae for the particular case of homogeneous four-dimensional spaces can be found in [IM98]. Here, we shall deal with much more general hypotheses.

Theorem 2.7 *Within our general hypotheses on $\mathcal{M}$ and A', and supposing $r > 0$, let λ be the first nonvanishing eigenvalue of A. For any real δm^2 such that $0 < \delta m^2 < \lambda$ and any integer $K > 0$, posing $B' := A' + \delta m^2 I$, one has*

$$\begin{aligned}\zeta(s,x|B/\mu^2) \;&=\; \left(\frac{\mu^2}{\delta m^2}\right)^s P_0(x,x) + \sum_{n=0}^{K}\left(-\frac{\delta m^2}{\mu^2}\right)^n \frac{\Gamma(s+n)}{n!\Gamma(s)}\zeta(s+n,x|A/\mu^2)\\ &\quad + \sum_{n=K+1}^{+\infty}\left(-\frac{\delta m^2}{\mu^2}\right)^n \frac{\Gamma(s+n)}{n!\Gamma(s)}\zeta(s+n,x|A/\mu^2)\,,\end{aligned} \tag{86}$$

where $x \in \mathcal{M}$ is fixed and Re $s \in [D/2-K, +\infty)$. Furthermore, the convergence of the series is uniform in any set Re $s \in [D/2-K, \beta]$ for any real $\beta > D/2-K$.

Proof. See **Appendix.** □

There is a trivial corollary of the theorem above concerning the field fluctuations. Indeed, we have for $Z(s,x|A/\mu^2)$ by (64)

$$\begin{aligned} Z(s,x|A+\delta m^2 I)/\mu^2) &= \frac{s}{\delta m^2}\left(\frac{\mu^2}{\delta m^2}\right)^s P_0(x,x) + Z(s,x|A/\mu^2) \\ &+ \sum_{n=1}^{K}\left(-\frac{\delta m^2}{\mu^2}\right)^n \frac{s\Gamma(s+1+n)}{\mu^2\Gamma(n+1)\Gamma(s+1)}\zeta(s+n+1,x|A/\mu^2) \\ &+ \sum_{n=K+1}^{+\infty}\left(-\frac{\delta m^2}{\mu^2}\right)^n \frac{s\Gamma(s+n+1)}{\mu^2\Gamma(n+1)\Gamma(s+1)}\zeta(s+n+1,x|A/\mu^2)\,. \end{aligned}$$

Notice that we can take the s derivative of this identity for $s=0$ passing the derivative under the symbol of series because each term of the series above is analytic and the series converges uniformly provided $K > D/2-1$ (the -1 is due to the evaluation of $\zeta(s,x)$ for $s+1$ in order to get $Z(s,x)$).
By (63) we have

$$\begin{aligned} \langle\phi^2(x|A+\delta m^2 I)\rangle_{\mu^2} &= \langle\phi^2(x|A)\rangle_{\mu^2} + \frac{P_0(x,x)}{\delta m^2} \\ &+ \sum_{n=1}^{+\infty}\left(-\frac{\delta m^2}{\mu^2}\right)^n \left[\frac{s\Gamma(s+1+n)}{\mu^2\Gamma(n+1)\Gamma(s+1)}\zeta(s+n+1,x|A/\mu^2)\right]'_{s=0} \end{aligned}$$

where the prime means the s derivative. By the point (b) of **Theorem 2.2** and the point (c) of **Theorem 2.5** we can rewrite the formula above in a improved form.

Theorem 2.8. *In the same hypotheses of* **Theorem 2.7** *the field fluctuations evaluated via local ζ function approach for the operator A and $A+\delta m^2 I$ are related by the relation (n is integer)*

$$\begin{aligned} \langle\phi^2(x|A+\delta m^2 I)\rangle_{\mu^2} &= \langle\phi^2(x|A)\rangle_{\mu^2} + \sum_{1\leq n\leq D/2-1}(-\delta m^2)^n\Phi_n(x|A)_{\mu^2} \\ &+ \sum_{n>D/2-1}(-\delta m^2)^n G(x,x|A^{n+1})\frac{P_0(x,x)}{\delta m^2}\,. \end{aligned} \tag{87}$$

where, if D is odd $\Phi_n(x|A)_{\mu^2}$ does not depend on μ^2 and

$$\Phi_n(x|A)_{\mu^2} = \mu^{-2(n+1)}\zeta(n+1,x|A/\mu^2) = \zeta(n+1,x|A) \tag{88}$$

and, if D is even

$$\Phi_n(x|A)_{\mu^2} = \mu^{-2(n+1)}\left[\frac{s\Gamma(s+1+n)}{\Gamma(n+1)\Gamma(s+1)}\zeta(s+n+1,x|A/\mu^2)\right]'_{s=0} \tag{89}$$

Notice that $\Phi_n(x|A)_{\mu^2}$ is always well-defined due to the meromorphic structure of the local zeta function which involves simple poles only. Moreover, (87) holds true also in the case P_0 does not vanish and thus A^{-1} does not exit. In this case the "Green functions" $G(x,y|A^{n+1})$ are not uniquely determined and are those defined via ζ by (71).

The found relation is the mathematically correct *local* form, in closed manifolds, of a formal relation assumed by physicists [Ev98], namely,

$$\int_{\mathcal{M}} \langle \phi^2(x|A+\delta m^2 I)\rangle_{\mu^2} d\mu_g(x) \;=\; \int_{\mathcal{M}} \langle \phi^2(x|A)\rangle_{\mu^2} d\mu_g(x) + \sum_{n=1}^{+\infty} (-\delta m^2)^n \mathrm{Tr} A^{-(n+1)} \,. \quad (90)$$

To get (90) one starts from the *correct* expansion holding for $|\delta m^2| < ||A^{-1}||^{-1}$ (provided A^{-1} exists)

$$(A+\delta m^2 I)^{-1} = A^{-1} + \sum_{n=1}^{+\infty} (-\delta m^2)^n A^{-(n+1)} \quad (91)$$

and uses the linearity of the trace operation and the generally *incorrect* identities ($n = 1, 2, \ldots$)

$$\langle \phi^2(x|A)\rangle = G(x,x|A) \qquad \text{and} \qquad \mathrm{Tr} A^{-n} = \int_{\mathcal{M}} G(x,x|A^n) d\mu_g(x)$$

The identities above do not hold in every cases as pointed out previously. In particular, barring trivial cases, the physically relevant dimension $D = 4$ generally involves the failure of both the identities above. Actually, the former identity never holds for $D > 1$. The latter generally does not hold for $n \leq D/2$ because A^{-n} is not a trace class operator. For $D = 4$, problems arise for the term $n = 1$ in (90). The task of the first sum in the right hand side of (87) is just to regularize the failure of the second identity above. In the case $D = 4$, (87) reads

$$\begin{aligned} \langle \phi^2(x|A+\delta m^2 I)\rangle_{\mu^2} \;&=\; \langle \phi^2(x|A)\rangle_{\mu^2} + \frac{P_0(x,x)}{\delta m^2} + (-\delta m^2)\Phi_1(x|A)_{\mu^2} \\ &+\; \sum_{n>1} (-\delta m^2)^n G(x,x|A^{n+1}) \,. \end{aligned} \quad (92)$$

We have also, after trivial calculations

$$\Phi_1(x|A)_{\mu^2} = [s\zeta(s+2,x|A)]_{s=0} + \mu^{-4} \left[s\zeta(s+2,x|A/\mu^2)\right]'_{s=0} \,. \quad (93)$$

A final remark for $D = 4$ is that, as one can prove directly

$$\int_{\mathcal{M}} [s\zeta(s+2,x|A)]_{s=0}\, d\mu_g(x) = \left[\int_{\mathcal{M}} s\zeta(s+2,x|A) d\mu_g(x)\right]_{s=0} = [s\zeta(s+2|A)]_{s=0} \quad (94)$$

In the case $D = 4$ and $P_0 \equiv 0$, one can check that A^{-2} is Hilbert-Schmidt and thus compact, moreover it is not trace-class but it belongs to $\mathcal{L}^{1+}$, the Macaev ideal [Co88]. Let us further suppose that A is a pure Laplacian. Then, the last term in the right hand side of (94) is nothing but the Wodzicki residue of A^{-2} [Wo84, Co88]. In other words, in the considered case, the last

term in the right hand side of (94) is four times the Dixmier trace of A^{-2} [Di66] because of a known theorem by Connes [Co88, El97].

We conclude this section noticing that, differently to that argued in [Ev98], not only the local ζ function approach is consistent, but it also agrees with the point-splitting procedure and it is able to regularize and give a mathematically sensible meaning to formal identities handled by physicists[6].

3 Summary and outlooks.

In this paper, we have proven that the local ζ function technique is rigorously founded and produces essentially the same results of the point-splitting, at least considering the effective Lagrangian and the field fluctuations. This result holds for any dimension $D > 1$ and in closed manifolds for Friedrichs extensions A of Schrödinger-like real positive smooth operator A'. Since these results are *local* results, we expect that this agreement does hold also dropping the hypothesis of a compact without boundary manifold. Several comments toward this generalization have been given throughout the paper.

Differences between the two approaches arise in the case of a untrivial $KerA$, when the local ζ function approach can be successfully employed whereas the point-splitting procedure is not completely well-defined.

Another results obtained in this paper is that the two-point functions, namely the Green function of A which we have built up via local ζ function and which is unique provided $P_0 \equiv 0$, has the Hadamard behaviour for short distance of the arguments for any $D > 1$. This fact allows the substantial equivalence of the two methods concerning the field fluctuation regularization. The only difference between the two approaches consists of the different freedom/ambiguity in choosing the term $w_0(x, y)$ of the Hadamard local solution.

Finally, we have discussed and rigorously proven a particular formula concerning the field fluctuations within our approach, proving that the ζ function procedure is able to regularize an identity which is supposed true by physicists but involves some mathematical problems when one tries to give rigorous interpretations of it .

An important issue which remains to be investigated is the equivalence of the local ζ-function approach and the point-splitting one concerning the one-loop stress tensor. This is an intriguing question also because the following weird reason. The point-splitting approach does not work completely in its naive formulation, as pointed out in [Wa94] (see also [BD82, Fu91]), at least in the case of a massless scalar field. In this case one cannot use Schwinger-DeWitt algorithm to pick out the term w_0 in the Hadamard expansion and, putting $w_0 \equiv 0$ one has to adjust by hand the final result to get either the conservation of the obtained stress tensor and, in the case of conformal coupling, the appearance of the conformal anomaly. Actually, this drawback does not arise within the local ζ function approach, as pointed out in [Mo97a], because the method

[6] This reply concerns only a part of criticism developed in [Ev98]. Several other papers (e.g. see [EFVZ98]) have recently appeared to reply to the objections aganist the multiplicative anomaly.

does not distinguish between different values of the mass and the coupling with the curvature.

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Appendix: Proof of some theorems.

Proof of Theorem 2.2. The idea is to break off the integration in (27) for $Re\ s > D/2$ as

$$\zeta(s,x,y|A/\mu^2) = \frac{\mu^{2s}}{\Gamma(s)}\int_0^{+\infty} dt\, t^{s-1}\left[K(t,x,y|A) - P_0(x,y|A)\right] \tag{95}$$

$$= \frac{\mu^{2s}}{\Gamma(s)}\int_0^{\mu_0^{-2}}\{\dots\} + \frac{\mu^{2s}}{\Gamma(s)}\int_{\mu_0^{-2}}^{+\infty}\{\dots\}\ , \tag{96}$$

where $\mu_0 > 0$ is an arbitrary mass cutoff.
We study the properties of these integrals separately. Let first focus attention on the second integral in right hand side of (96) considered as a function of $s \in \mathbb{C}$, $x, y \in \mathcal{M}$.
From **Theorem 1.1**, and using Cauchy-Schwarz inequality, one finds straightforwardly

$$|K(t,x,y|A) - P_0(x,y|A)|^2 \leq [K(t,x,x|A) - P_0(x,x|A)][K(t,y,y|A) - P_0(y,y|A)]\ . \tag{97}$$

Moreover, let us define

$$p(t,x) := e^{\lambda t}[K(t,x,x|A) - P_0(x,x|A)] \geq 0 \tag{98}$$

λ being the first strictly positive eigenvalue of A. From the expansion (15) it is obvious that, since $\lambda_j - \lambda \geq 0$ whenever $\lambda_j \neq 0$, $p(t,x) \geq p(t',x)$ for $t' \geq t$ and thus, in $[\mu_0^{-2}, +\infty)$, $p(t,x) \leq p(\mu_0^{-2}, x)$. This last function is continuous in $\mathcal{M}$ by construction (essentially, it is $K(t,x,x)$ self). Hence, $\max_{x\in\mathcal{M}} p(\mu_0^{-2}, x)$ does exist because the compactness of $\mathcal{M}$. From (98) one has the (x,y)-uniform upper bound of the heat kernel in $t \in [\mu_0^{-2}, +\infty)$

$$|K(t,x,y|A) - P_0(x,y|A)| \leq \max_{z\in\mathcal{M}}[K(\mu_0^{-2}, z, z|A) - P_0(z,z|A)]\ e^{-\lambda(t-\mu_0^{-2})} \tag{99}$$

This result proves that the second integral in (96) converges absolutely, not depending on $s \in \mathbb{C}$, and the final function is in $C^0(\mathcal{M}\times\mathcal{M})$ for any fixed $s \in \mathbb{C}$ because the upper bound in (99) does not depend on x, y. Actually, studying the function $t \mapsto t^s \exp(-\lambda t)$, one finds that, for $\mu_0 > 0, \lambda > 0, \epsilon \in (0,1)$ there is a positive constant $B = B(\mu_0^{-2}, \lambda, \epsilon)$ such that, for $s \in \mathbb{C}$, $|t^s \exp(-\lambda t)| \leq \exp[B\ (Re\ s)^2]\exp(-\epsilon\lambda t)$ whenever $t \in [\mu_0^2, +\infty)$ and thus one has, for t belonging to that interval and $Re\ s \in [\alpha, \beta]$, $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$

$$|t^s e^{(-\lambda t)}| \leq \left(e^{B\alpha^2} + e^{B\beta^2}\right) e^{-\epsilon\lambda}\ . \tag{100}$$

This implies that the considered integral defines also a function of s, x, y which belongs to $C^0(\mathbb{C} \times \mathcal{M} \times \mathcal{M})$. The same bound (the s derivatives produces integrable factors $\ln t$ in the integrand) proves that the considered integral is also an analytic function of $s \in \mathbb{C}$ for x, y fixed in $\mathcal{M}$. This is because, by Lebesgue's dominate convergence theorem, one can interchange the symbol of t integration with the derivative in $Re\ s$ and $Im\ s$ and prove Cauchy-Riemann's identities. Moreover, by the same way, it is trivially proven that all of s derivatives of the considered integral belong to $C^0(\mathbb{C} \times \mathcal{M} \times \mathcal{M})$.
The remaining integral in (96) needs further manipulations. Using **Theorem 1.3** and (19) for any integer $N > D/2 + 2$, it is now convenient to consider the function

$$\begin{aligned}\zeta(N, s, x, y|A/\mu^2, \mu_0^{-2}) \quad &:= \quad \frac{\mu^{2s}}{\Gamma(s)} \int_{\mu_0^{-2}}^{+\infty} dt\, t^{s-1} \left[K(t, x, y|A) - P_0(x, y|A)\right] \\ &+ \frac{\mu^{2s}}{\Gamma(s)} \int_0^{\mu_0^{-2}} dt\, t^{s-1} \frac{e^{-\eta\sigma(x,y)/2t}}{(4\pi t)^{D/2}} t^N O_\eta(t; x, y)\,. \end{aligned} \tag{101}$$

Let us study this function for a fixed $N > D/2 + 2$. For $Re\ s \in [D/2 - N + \epsilon, \beta]$, β being any real $> D/2 - N + \epsilon$ and $\epsilon > 0$ another real, because $O_\eta(t, x, y)$ is bounded we have the following (s, x, y)-uniform bound of the integrand in the second integral in (101)

$$|e^{-\eta\sigma(x,y)/2t} t^{s-1+N-D/2} O_\eta(t; x, y)| \leq K t^{\epsilon-1} \chi_1(t) + K \mu_0^{-2|\beta+N-D/2-1|} \chi_2(t)\,, \tag{102}$$

where $K > 0$ is a constant not depending on x and y, χ_1 is the characteristic function of the set $[0, \min\{1, \mu_0^{-2}\}]$ and χ_2 the characteristic function of the set $[\min\{1, \mu_0^{-2}\}, \mu_0^{-2}]$ ($\chi_2 \equiv 0$ if this set is empty). As before, this result implies that the second integral in (101) and all of its s derivatives, considered as a function of s, x, y, belong to $C^0(\{s \in \mathbb{C} \mid Re\ s > D/2 - N\} \times \mathcal{M} \times \mathcal{M})$. Moreover, for x, y fixed, this function is analytic in $\{s \in \mathbb{C} \mid Re\ s > D/2 - N\}$.
Notice that $\zeta(N, 0, x, y|A/\mu^2, \mu_0^{-2}) = 0$ for any integer $N > D/2 + 2$ and $x, y \in \mathcal{M}$.
If the points x and y do not coincide the exponential function in the right hand side of (101) sharply decays as $t \to 0^+$. In fact, for $(x, y) \in \mathcal{G}$ where $\mathcal{G}$ is any compact subset of $\mathcal{M} \times \mathcal{M}$ which does not contain elements of the form (x, x), we have the following (s, x, y)-uniform bound for $Re\ s \in [\alpha, \beta]$ for any choice of $\alpha < \beta$ in $\mathbb{R}$

$$\begin{aligned}|e^{-\eta\sigma(x,y)/2t} t^{s-1+N-D/2} O_\eta(t; x, y)| \quad &\leq \quad e^{-\eta\sigma_0/2t} K' t^{\alpha+N-D/2-1} \chi_1(t) \\ &+ K' e^{-\eta\sigma_0/2t} \mu_0^{-2|\beta+N-D/2-1|} \chi_2(t)\,, \end{aligned} \tag{103}$$

where $K' > 0$ is a constant not depending on x and y and $\sigma_0 = \min_{\mathcal{G}} \sigma(x, y)$ which is strictly positive. Notice that no limitations appears on the choice of $[\alpha, \beta]$ namely, in the range of s.
Therefore, by the same way followed in the general case we have that $\zeta(N, s, x, y|A/\mu^2, \mu_0^{-2})$ defines a function which belongs, together with all of s derivatives, to $C^0(\mathbb{C} \times ((\mathcal{M} \times \mathcal{M}) - \mathcal{D}))$ where $\mathcal{D} := \{(x, y) \in \mathcal{M} \times \mathcal{M} \mid x = y\}$.
Coming back to the local ζ function, by the given definition we have that, for $N > D/2 + 2$

$$\zeta(s, x, y|A/\mu^2) \quad = \quad \zeta(N, s, x, y|A/\mu^2, \mu_0^{-2}) - \frac{\mu^{2s} P_0(x, y)}{\Gamma(s)} \int_0^{\mu_0^{-2}} dt\, t^{s-1}$$

$$+\frac{\mu^{2s}}{\Gamma(s)}\int_0^{\mu_0^{-2}} dt\, t^{s-1}\frac{e^{-\sigma(x,y)/2t}}{(4\pi t)^{D/2}}\chi(\sigma(x,y))\sum_{j=0}^{N} a_j(x,y|A)t^j \qquad (104)$$

Up to now we have proven that, for x, y fixed, the first term in the right hand side can be analytically continued (actually is directly computable there) at least in the set $\{s \in \mathbb{C} \mid Res > D/2 - N\}$ and furthermore everywhere for $x \neq y$, moreover, varying also x and y one gets a (at least) (s, x, y)-continuous function also considering the s derivatives. This function vanishes for $s = 0$.

The second term in the right hand side of (104) can be computed for $Re\ s > 0$ and then the result can be continued in the whole s complex plane defining a s-analytic function $C^\infty(\mathbb{C} \times \mathcal{M} \times \mathcal{M})$. This function gets the value $-P(x, y)$ for $s = 0$. We can rearrange (104) after the analytic continuation of the term containing P_0 as

$$\begin{aligned}\zeta(s,x,y|A/\mu^2) &= \zeta(N,s,x,y|A/\mu^2,\mu_0^{-2}) - \left(\frac{\mu}{\mu_0}\right)^{2s}\frac{P_0(x,y)}{s\Gamma(s)} \\ &+\frac{\mu^{2s}\chi(\sigma(x,y))}{(4\pi)^{D/2}\Gamma(s)}\sum_{j=0}^{N} a_j(x,y|A)\int_0^{\mu_0^{-2}} dt\, t^{s-1+j-D/2}e^{-\sigma(x,y)/2t} \qquad (105)\end{aligned}$$

This can be considered as another definition of $\zeta(N, s, x, y|A/\mu^2, \mu_0^{-2})$ equivalent to (101) in the sense of the s analytic continuation. Concerning the last term of the right hand side of (105) we have to distinguish between two cases.

For $x \neq y$, following procedures similar to those above, it is quite simply proven that the last term in the right hand side defines an everywhere s-analytic function $C^\infty(\mathbb{C} \times ((\mathcal{M} \times \mathcal{M}) - \mathcal{D}))$ as it stands. Once again, this result is achieved essentially because of the sharp decay of the exponential as $t \to 0^+$. We notice also that the considered term vanishes for $s = 0$.

Summarizing, in the case $x \neq y$, the left hand side of (104) defines an everywhere s analytic function which, at least, belongs also to $C^0(\mathbb{C} \times ((\mathcal{M} \times \mathcal{M}) - \mathcal{D}))$ together with all of its s derivatives. Moreover it vanishes for $s = 0$ giving rise to (34) in the case $x \neq y$. The order of the zero at $s = 0$ in the right hand side of (34) is at least 1 because of the overall factor $1/\Gamma(s)$ in (105). Up to now, we have proven $(a1), (a2)$ and (c) partly.

Let us finally consider the last term in the right hand side of (104) in the case $x = y$. In this case we cannot take advantage of the sharp decay of the exponential. However, we can perform the integration for $Re\ s > D/2$ and then continue the result as far as it is possible in the remaining part of the s-complex plane. Notice that, away from the poles, the obtained function is C^∞ in s, x, y trivially. We have finally

$$\begin{aligned}\zeta(s,x|A/\mu^2) &= \zeta(N,s,x,x|A/\mu^2,\mu_0^{-2}) - \frac{(\mu/\mu_0)^{2s}P_0(x,x)}{s\Gamma(s)} \\ &+\frac{\mu^{2s}}{(4\pi)^{D/2}}\sum_{j=0}^{N}\frac{a_j(x,x|A)(\mu_0^{-2})^{(s+j-D/2)}}{\Gamma(s)(s+j-D/2)}\,. \qquad (106)\end{aligned}$$

This identity defines an analytic continuation $\zeta(s, x|A/\mu^2)$ at least in $\mathcal{D}_N = \{s \in \mathbb{C}|Re\ s > D/2 - N\}$ for each integer $N > D/2 + 2$, indeed, therein both functions (and all of their s

derivatives) in right hand side are defined and continuous in (x,s) away from the possible poles. Summarizing, the left hand side of the equation above is decomposed into a function analytic in $\mathcal{D}_N$ and a function which is meromorphic in the same set, both functions and their s derivatives are at least continuous in (s,x) away from possible poles. Noticing that $\mathcal{D}_N \subset \mathcal{D}_{N+1}$ and $\bigcup_{N=1}^{+\infty} \mathcal{D}_N = \mathbb{C}$, the properties found out for the function $\zeta(s,x,y|A/\mu^2)$ can be extended in the whole s-complex plane. In particular, the continued local ζ function and all of its s derivatives belong to $C^0((\mathbb{C}-\mathcal{P})\times\mathcal{M})$ at least, where $\mathcal{P}$ is the set of the actual poles of the last term in the right hand side of (106). Notice that, for $s=0$, (106) gives (34) in the case $x=y$. This proves (b) and complete the proof of (c).
The proof of the part (d) of the theorem concerning the integrated ζ function is very similar to the case $x=y$ treated above. And one straightforwardly finds that the s-continuation procedure commutes with the integration procedure of the local (on-diagonal) ζ function as a consequence of the Fubini theorem. $\Box$

Proof of Theorem 2.7. In our hypotheses, $Ker\ B=\{0\}$ and thus the local ζ function of B is defined as in (27) for $x=y$, $Re\ s>D/2$ and without the term $P(x,x|B)$ in the integrand. The expression of $K(t,x,x|B)$ is very simple, in fact one has

$$K(t,x,y|B) = e^{-\delta m^2 t} K(t,x,y|A)\,. \tag{107}$$

Indeed, the right hand side satisfies trivially the heat equation (12) for B whenever $K(t,x,y|A)$ satisfies that equation for A, and this must be the only solution because of **Theorem 1.1**. Therefore, we have

$$\zeta(s,x|B/\mu^2) = z_1(s,x)+z_2(s,x)+z_3(s,x) \tag{108}$$

where, we have decomposed the right hand side in a sum of three parts after we have taken the expansion of the exponential $\exp -\delta m^2 t$. For $Re\ s>D/2$ they are, P_0 being the projector on the kernel of A and $\mu_0>0$ any real constant,

$$z_1(s,x) := \frac{\mu^{2s}}{\Gamma(s)}\int_{\mu_0^{-2}}^{+\infty} dt\, \sum_{n=0}^{+\infty}\frac{(-\delta m^2)^n}{n!} t^{s-1+n}[K(t,x,x|A)-P_0(x,x)] \tag{109}$$

$$z_2(s,x) := \frac{\mu^{2s}}{\Gamma(s)}\int_0^{\mu_0^{-2}} dt\, \sum_{n=0}^{+\infty}\frac{(-\delta m^2)^n}{n!} t^{s-1+n}[K(t,x,x|A)-P_0(x,x)] \tag{110}$$

$$z_3(s,x) := \left(\frac{\mu^2}{\delta m^2}\right)^s P_0(x,x)\,. \tag{111}$$

The last term is the result of

$$\frac{\mu^{2s}}{\Gamma(s)}\int_0^{+\infty} dt\, e^{-\delta m^2 t} t^{s-1} P_0(x,x) = \left(\frac{\mu^2}{\delta m^2}\right)^s P_0(x,x)\,.$$

We want to study separately the s analytic continuations of the first two terms above. The last term does not involves particular problems. In particular we shall focus attention on the

possibility of interchange the sum with the integral and we want to discuss the nature of the convergence of the series once one has interchanged the sum over n with the integral over t. In fact, we want to prove that the convergence is uniform in s within opportune sets. Let us start with $z_1(s,x)$. We consider the double integral in the measure "n-sum $\otimes$ t-integral" of nonnegative elements

$$I := \sum_{n=0}^{+\infty} \int_{\mu_0^{-2}}^{+\infty} dt\, \frac{(\delta m^2)^n}{n!} |t^{s-1+n}[K(t,x,x|A) - P_0(x,x)]| \tag{112}$$

for $Re\ s \in [\alpha, \beta]$, $\alpha, \beta \in I\!R$ fixed arbitrarily, $t \in [\mu_0^{-2}, +\infty)$. Following the same procedure than in the proof of **Theorem 2.2**, from (100) we have that, for any $\epsilon \in (0,1)$, there is a positive B, such as, in the sets defined above

$$\frac{(\delta m^2)^n}{n!} \int_{\mu_0^{-2}}^{+\infty} dt\, |t^{s-1+n}||K(t,x,x|A) - P_0(x,x)| \leq a_n\ , \tag{113}$$

where

$$a_n := \frac{(\delta m^2)^n}{n!} \int_{\mu_0^{-2}}^{+\infty} dt\, t^n e^{-\epsilon\lambda t} \left(e^{B(\alpha-1)^2} + e^{B(\beta-1)^2}\right)\ . \tag{114}$$

The series of the positive terms a_n converges provided $\epsilon \in (0,1)$ is chosen to give $\epsilon\lambda > \delta m^2$, the sum of a_n being bounded by the t integral, in $[\mu_0^2, +\infty)$, of the function

$$t \mapsto \left(e^{B(\alpha-1)^2} + e^{B(\beta-1)^2}\right) e^{-(\epsilon\lambda - \delta m^2)t} \tag{115}$$

Therefore, by a part of Fubini's theorem, we have proven that the function defined for $Re\ s$ fixed in $[\alpha,\beta]$, $t \in [\mu_0^2, +\infty)$, $n \in I\!N$

$$(n,t) \mapsto \frac{(-\delta m^2)^n}{n!} t^{s-1+n}[K(t,x,x|A) - P_0(x,x)] \tag{116}$$

is integrable in the product measure above, and thus, again by Fubini's theorem, we can interchange the series with the integrals in (109). Moreover the series of integrals so obtained is s-uniformly convergent in $Re\ s \in [\alpha,\beta]$ because this series is term-by-term bounded by the series of the positive numbers a_n which do not depend on s. The s-analyticity, for $s \in (\alpha,\beta)$, of the terms of the series of the integrals was still proven in the proof of **Theorem 2.2**. $z_1(s,x)$ defines an analytic function in any set $Re\ s \in (\alpha,\beta)$ and it can be computed by summing the series of analytic functions, uniformly convergent

$$z_1(s,x) = \sum_{n=0}^{+\infty} \frac{(-\delta m^2)^n}{n!} \frac{\mu^{2s}}{\Gamma(s)} \int_{\mu_0^{-2}}^{+\infty} dt\, t^{s-1+n}[K(t,x,x|A) - P_0(x,x)] \tag{117}$$

Let us consider $z_2(s,x)$. By **Theorem 1.4** for $Re\ s > D/2$, we can decompose

$$\begin{aligned} z_2(s,x) &= \frac{\mu^{2s}}{\Gamma(s)} \int_0^{\mu_0^{-2}} dt \sum_{n=0}^{+\infty} \frac{(-\delta m^2)^n}{n!} t^{s-1+n-D/2+N} O_\eta(t;x,x) \\ &+ \sum_{j=0}^{N} a'_j(x,x|A) \frac{\mu^{2s}}{\Gamma(s)} \int_0^{\mu_0^{-2}} dt \sum_{n=0}^{+\infty} \frac{(-\delta m^2)^n}{n!} t^{s-1+n-D/2+j}\,. \end{aligned} \tag{118}$$

Above $a'_j(x,x|A) = a'_j(x,x|A)/(4\pi)^{D/2}$ for $j \neq D/2$ and $a'_{D/2}(x,x|A) = a'_{D/2}(x,x|A)/(4\pi)^{D/2} - P_0(x,x)$ otherwise, moreover $N > D/2+2$ is any positive integer, and $O_\eta(s;x,y)$ was defined in the cited theorem (we have omitted the constant factor $(4\pi)^{-D/2}$).
Let us consider the first integral in the right hand side above. It is possible to prove that, in any set $Re\ s \in [D/2-N+\epsilon, \beta]$, $\beta \in I\!R$ ($N \geq D/2$ integer and $\epsilon > 0$ are fixed arbitrarily and $D/2-N+\epsilon < \beta$) one can interchange the symbol of integration with that of series. Moreover the terms of the new series are analytic function of s in $Re\ s \in (D/2-N+\epsilon, \beta)$ and the new series converges s uniformly in the considered closed set.
The proof is very similar to the proof of the analogue statement for $z_1(s,x)$. The the possibility to interchange the integral with the series follows from Fubini's theorem for the measure "n-sum $\otimes$ t-integration" exactly as in the previous case, and employs the uniform bound, obtained from (102) within the proof of **Theorem 2.2**

$$\int_0^{\mu_0^{-2}} dt\, |t^{s-1+n-D/2} O_\eta(t;x;x)| \leq C \frac{\mu_0^{-2(n+\epsilon)}}{n+\epsilon} + C \frac{\mu_0^{-2(n+1)}}{n+1} \mu_0^{-2|\beta+N-D/2-1|} \tag{119}$$

(C being a positive constant). Notice also that, trivially,

$$\sum_{n=0}^{+\infty} \frac{(\delta m^2)^n}{n!} \left(\frac{\mu_0^{-2(n+\epsilon)}}{n+\epsilon} + \frac{\mu_0^{-2(n+1)}}{n+1} \mu_0^{-2|\beta+N-D/2-1|} \right) < +\infty \tag{120}$$

The analyticity of the terms in the series of the integrals deal with as pointed out in the proof of **Theorem 2.2**. The uniform convergence of the series follows from the s-uniform bound (119). Hence, the first term in the right hand side of (118) can be written as

$$z'_2(s,x) = \frac{\mu^{2s}}{\Gamma(s)} \sum_{n=0}^{+\infty} \frac{(-\delta m^2)^n}{n!} \int_0^{\mu_0^{-2}} dt\, t^{s-1+n-D/2+N} O_\eta(t;x,x). \tag{121}$$

It defines an analytic function in any set $Re\ s \in [D/2-N+\epsilon, \beta]$ and the convergence of the series is uniform therein. Notice that N as well as β are arbitrary and thus the set above can be enlarged arbitrarily.

Finally let us consider the remaining term in the right hand side of (118) initially defined for $Re\ s > D/2$

$$z''_2(s,x) = \sum_{j=0}^{N} a'_j(x,x|A) \frac{\mu^{2s}}{\Gamma(s)} \int_0^{\mu_0^{-2}} dt \sum_{n=0}^{+\infty} \frac{(-\delta m^2)^n}{n!} t^{s-1+n-D/2+j}\,. \tag{122}$$

We can decompose the sum over n into two parts

$$
\begin{aligned}
z_2''(s,x) &= \sum_{n=0}^{K}\frac{(-\delta m^2)^n}{n!}\sum_{j=0}^{N}a_j'(x,x|A)\frac{\mu^{2s}}{\Gamma(s)}\int_0^{\mu_0^{-2}}dt\, t^{s-1+n-D/2+j} \\
&+ \sum_{j=0}^{N}a_j'(x,x|A)\frac{\mu^{2s}}{\Gamma(s)}\int_0^{\mu_0^{-2}}dt\sum_{n=K+1}^{+\infty}\frac{(-\delta m^2)^n}{n!}t^{s-1+n-D/2+j}\,.
\end{aligned}
\tag{123}
$$

The first part can be continued in the whole s complex plane away from the poles obtained by executing the integrals. The second part can be studied as in the cases discussed above. One has to study the convergence of

$$
I' := \sum_{n=K+1}^{+\infty}\frac{(\delta m^2)^n}{n!}\sum_{j=0}^{N}|a_j'(x,x|A)|\int_0^{\mu_0^{-2}}dt\,|t^{s-1+n-D/2+j}|\,.
\tag{124}
$$

Let us consider $Re\ s \in [D/2-K,\beta]$, where $\beta \in I\!R$ is arbitrary. Let us pose

$$
M := \sup_{s\in[D/2-K,\beta],0\leq j\leq N}\left\{|a_j'(x,x|A)|\mu_0^{-2(Res-D/2+j)}\right\}\,,
\tag{125}
$$

then we have (notice that $n = K+1, K+2, \ldots$)

$$
\frac{(\delta m^2)^n}{n!}\sum_{j}^{N}|a_j'(x,x|A)|\int_0^{\mu_0^{-2}}dt\,|t^{s-1+n-D/2+j}| \leq \frac{(\delta m^2)^n}{n!}\frac{(N+1)M\mu_0^{-2n}}{n-K} =: b_n\,.
\tag{126}
$$

The series of the positive coefficients b_n converges trivially. As in the cases previously treated, this is sufficient to assure the possibility to interchange the sum over n with the t integration in the second line of (123) and assure the s-uniform convergence of the consequent series in *Re* $s \in [D/2-K,\beta]$. This completes the proof of the theorem. □.

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